THE EXTENDED RULE OF MAXWELL AND RIGIDITY CONDITIONS FOR INFINITESIMAL MECHANISMS

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Abstract: *The present paper is concerned with the analysis of the static and kinematic behaviour of pin-jointed frameworks. In order to achieve the most correct analysis an extended formulation of Clerk Maxwell's rule for stiff frames is presented, which can provide us with useful information regarding the number of states of independent self-stresses and inextensional mechanisms. A brief explanation of mechanism theory is given. Finally prestress stability and rigidity conditions are set for first-order infinitesimal mechanisms.*

Key words: *Maxwell's rule, extended Maxwell's rule, infinitesimal mechanism, rigidity, Tensegrity.*

1. Introduction

A key feature to understanding the mechanics of any pin-jointed framework is the concept of static and/or kinematic determinacy of the analysed assembly.

Möbius in 1837 was the first to show the necessary conditions in order to render a bar framework stable. According to his findings a general pin-jointed plane framework consisting of n joints must have at least $2n - 3$ bars in order to be rigid, while a space framework needs at least $3n - 6$ [3].

Couple of decades later Maxwell in an attempt to introduce a new method to the analysis of bar tensions and nodal deflections of frameworks "*in the least complicated manner… especially in cases in which the framework is not simply stiff but is strengthened… by additional connecting pieces*" [4] came to the same results as Möbius. Although, at the time of publication, Maxwell's work failed to make a great impact, later his statement regarding the necessary conditions to render a frame simply stiff became the long-standing industry standard in the design of pin jointed frameworks under the name **Maxwell's Rule**. However, the appearance of novel type structures, such as Buckminster Fuller's Tensegrity frameworks, which are rigid without the existence of the necessary number of bars, make a compelling argument for extending Maxwell's original rule in order to adapt to the necessities of the newest discoveries in the field of civil engineering.

2. Maxwell's Rule for the Analysis of Pin-Jointed Structures

James Clerk Maxwell in the introductory part of his article "*On the Calculation of*

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the Equilibrum and Stiffness of Frames" enounced the famous statement, which became later known in the mechanics of frameworks as the rule bearing his name: "*A frame of s points in space requires in general 3s – 6 connecting lines to render it stiff*".

Mathematically Maxwell's rule for a simply stiff frame composed of n nodes, b bars and r simple restraints can be formulated in the following way:

$$3n \leq b + r. \tag{1}$$

Analysing the aforementioned formulation of Maxwell's rule - i.e. Eq. (1) - one can easily observe the existence of three separate cases, which correspond to the three fundamental kinematic behaviour of structures:

1. The case of kinematically indeterminate structures, where $3n > b + r$.

2. The case of simply stiff structures, characterized by kinematic and static determinacy ($3n = b + r$).

3. The case of statically indeterminate structures, when the structure has more than the necessary number of bars or restraints ($3n < b + r$).

Besides the guidance for the determination of the necessary number of bars and foundation joints Maxwell's rule provides another essential information for the static analysis of pin-jointed structures by showing the unique determinedness (or the lack of it) of the member forces for any arbitrary external loading.

As the years passed the simplicity of Maxwell's rule, being based only on a simple counting strategy of the comprising elements of a structure, quickly made it the most valuable tool for the engineering community in the analysis of static and kinematic behaviour of frameworks. Recently, because of the appearance of a special type of pin-jointed framework named Tensegrity, the original

formulation of Maxwell's rule went over a major change.

3. Tensegrity Structures and the Extended form of Maxwell's Rule

In an attempt to create an explanation for the mechanics of Robert Buckminster Fuller's Tensegrity structures Calladine [1] went back to Maxwell's paper. He analysed a Truncated Tensegrity Tetrahedron (Figure 1). The geometrically stable, stiff structure is made of 24 bars and 12 joints, although according to Maxwell's rule it would need 6 additional bars in order to be simple stiff [1]. Thus this frame constitutes a paradoxical exception to Maxwell's Rule.

The brilliance of James Clerk Maxwell can be shown by him expecting the existence of such "ill-conditioned" frames, which are defined by him as follows: "*In those cases where stiffness can be produced with a smaller number of lines, certain conditions must be fulfilled, rendering the case one of a maximum or a minimum value of one or more of its lines.*"
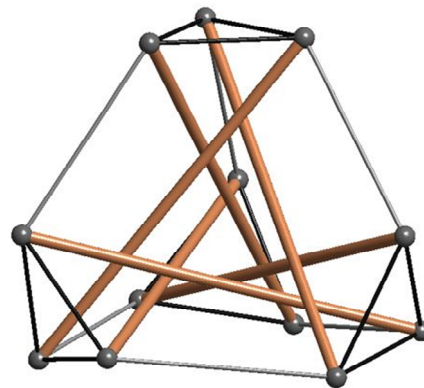

Fig. 1. *The truncated tetrahedron analysed by Calladine* [3]

The experimental analysis of Fuller's Tensegrity structure from Figure 1 showed that certain configurations of the assembly comprised of bars with

arbitrary length indeed act as a mechanism, but when the length of the bars reached a certain value (i.e. the maximal length mentioned by Maxwell) the structure became stiff. This behaviour clearly shows that there is a limit to the length of the bars and cables of Fuller's structures. Reaching this limit length of the comprising members, presumably the maximum value Maxwell had in mind, renders these structures stiff. This behaviour of Fuller's structures seems to match the one of the special cases anticipated by Maxwell.

Moreover Maxwell is aware that these exceptional structures essentially constitute a special configuration of kinematically indeterminate frames, which have limited stiffness, stating that the stiffness of such a frame *"is of an inferior order, as a small disturbing force may produce a displacement infinite in a comparison with itself"* [4].

3.1. Maxwell's Extended Rule

In order to illustrate the need for changing Maxwell's original rule let us analyse the two pin-jointed frames from Figure 2. They are made of the same number of bars ($b = 2$), joints ($n = 3$) and restraints ($r = 4$). Moreover according to Maxwell's Rule they constitute simple stiff structures by satisfying the equality ($2 \cdot 3 = 2 + 4$). However a closer look at the fundamental behaviour of these structures shows that the looks can be deceiving.

In the first configuration, Figure 2a), the frames are not collinear, hence the frame is rigid and capable to bear the nodal force *P*.

The second structure, Figure 2b), consists of three nodes which lie along the same line. It can be easily spotted that in the case of an external loading normal to the plane of junction this frame will suffer a vertical displacement *v*. The stiffness for vertical forces of such frame when the structure lacks pretension is proportional to the third power of the displacement of the central joint [1]. This value is very small, being in concordance with "the stiffness of an inferior order" noted by Maxwell, thus rendering the infinitesimal mechanism behaviour of the framework.

As noted in the case of Fuller's Tensegrity structures this framework can be rendered stiff by means of adequate pre-tensioning. The value of the necessary prestress needed to be applied to the bars yields from the vertical equilibrium at the central joint:

$$P \approx 4 \cdot t_0 \cdot \frac{v}{l}, \tag{2}$$

where t_0 is the prestressing force applied to the bars, *l* represents the length between the restrained joints of the framework.

Pre-tensioning the two-bar pin-jointed assembly with the value determined from Eq. (2) stabilises the internal mechanism, hence the structure will be able to resist the vertical loading *P*.
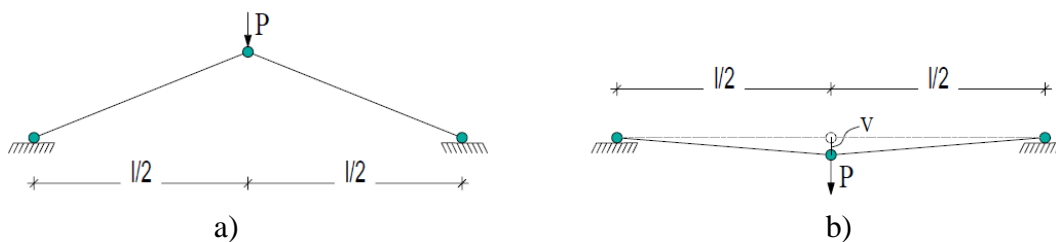

a)   b)

Fig. 2. *Pin-jointed Two-bar frameworks*

Calladine [1] and Pellegrino and Calladine [5] proposed a modified formulation for Maxwell's original rule which includes all kinds of special cases foreseen by Maxwell himself. According to the Extended Maxwell's Rule a complete analysis of a pin-jointed framework can be made only by considering besides the terms from the Maxwell's original Rule the number of the independent states of self-stresses *s* and the one of the inextensional mechanisms *m*:

$$3n + s \le b + r + m. \tag{3}$$

The number of the independent self-stresses and the number of infinitesimal mechanisms can be derived from the rank r_A of the equilibrium matrix A and the compatibility matrix A^T as follows:

$$m = 3n - r_A, \tag{4.1}$$

$$s = b + r - r_{A^T}. \tag{4.2}$$

It is worth to note that the values of *m* and *s* depend not only on the number of bars and joints, nor the topology of the structure, but on the complete definition of the geometry of the pin-jointed framework [5].

4. Finite Mechanism vs Infinitesimal Mechanism

For a better understanding of the behaviour of the kinematically indeterminate structures let us take a closer look at the mechanics of the pin-jointed frameworks from Figure 3.

Although both frames satisfy Maxwell's original rule they actually are statically and kinematically indeterminate structures, which allow for one state of prestress ($s = 1$) and one inextensional mechanism ($m = 1$). However their kinematic behaviour is *quantitatively very different*. The structure from Figure 3a has an *infinitesimal mechanism*, while the one from Figure 3b has a *finite mechanism*.

As shown in the aforementioned figure the common joint of the structure (O) from Figure 3a can move in the direction perpendicular to the plane of the bars. Essentially the comprising bars of the structure exhibit deformations which are of higher order in terms of the displacement of the joint O.

On the other hand, the second structure can suffer large displacement without the need of any variation in the length of the bars. Thus, the difference between the kinematic behaviour of the two frameworks
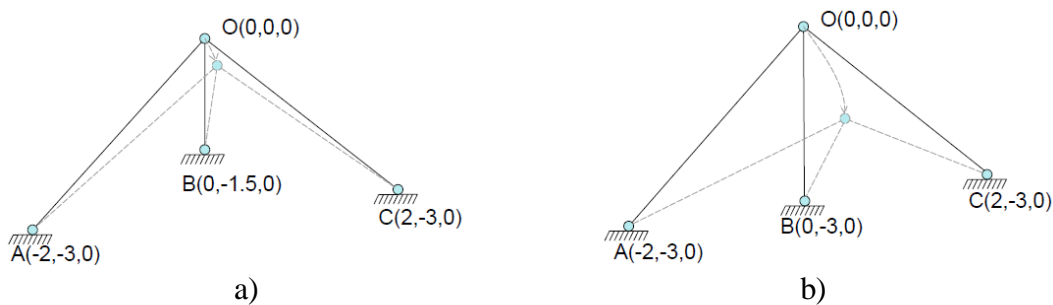

a)   b)

Fig. 3. *Statically and kinematically indeterminate structures: infinitesimal mechanism (a), finite mechanisms (b)*

can be easily spotted and understood. Essentially the movement of the central joint can be visualized by intersecting two spheres centered at *A* and *C* and with equal radii (*AO = CO*).

In the case of the second structure the resultant circle, with a radius of *BO*, belongs to the sphere centered at *B* and with radius equal to the length of the middle bar (*BO*). This characteristic does not prohibit the common joint *O* experiencing large movement, in a direction perpendicular to the plane of the frames (*OAC*), along the aforementioned circle. In conclusion this structure is a *finite mechanism*.

On the other hand in the case of the framework from Figure 3a) the displacement of the common joint *O* is limited by the fact that the sphere centered at node *B* is only tangent to the aforementioned two spheres centered at joints *A* and *C*. Thus only a very limited displacement of the common joint can be experienced by the structure, hence it has only one *infinitesimal mechanism*.

Unfortunately classifying the statically and kinematically indeterminate structures only into finite mechanisms and infinitesimal mechanisms is inadequate to deal with all the possible existing cases.

Koiter completed the theory of mechanisms by defining "*an infinitesimal mechanism of the first order by its property that any infinitesimal displacement of the mechanism is accompanied by second order elongations of at least some of the bars. An infinitesimal mechanism is called of second (or higher) order, if there exists an infinitesimal motion such that no bar undergoes an elongation of lower than the third (or higher) order*" [6].

The mathematical formulation of Koiter's definition has been provided by Tarnai in [6]. He analysed the system of infinitesimal displacements of the joints of a pin-jointed assembly which contains *b* bars. According to Tarnai the elongation of a bar (e_k) can be produced by the power series of the infinitesimal displacement δ of a characteristic joint:

$$e_k = a_{1k}\delta + a_{2k}\delta^2 + a_{3k}\delta^3 + ... \qquad (5)$$

where $k = 1, 2, …, b$.

Tarnai later completed Koiter's definition by stating: "*an infinitesimal mechanism is of order n ($n \geq 1$) if there exists a system of infinitesimal displacements of joints such that... $a_{1,k} = a_{2,k} = … = a_{n,k} = 0$ for $k = 1, 2, …, b$, but there exists no system of infinitesimal displacement of joints such that... $a_{n+1,k} = 0$ for $k = 1, 2, …, b$*" [6]. In other words there exists at least one bar *m* for which $a_{n+1,m} \neq 0$.

The definition of the finite mechansims can be formulated working along the line set by Tarnai's definition of higher order rigidity. Given that by their fundamental properties the bars of a finite mechansim have zero elongation it follows that a *finite mechanism can be defined as an infinitesimal mechanism of infinite order* [6].

5. Rigidity Conditions for First-Order Infinitesimal Mechanisms

One of the greatest problems the engineering community in our days is to create proper configurations of structures with the usage of less and less material. In order to conquer this great challenge more and more engineers try to capitalize the beneficial properties of tensioned structures. As a great number of these structures are statically and kinematically indeterminate the most important problem which needs to be solved is the answer to the following questions: When are the statically and kinematically indeterminate structures in equilibrium? What conditions need to be satisfied by a statically and kinematically indeterminate structure in order to be rendered stiff by the pre-tensioning forces?

Paradoxically the answer to both questions did not come from the work of some prominent engineers, but from the mathematical research of the tensegrity structures. The most important result of this research was the extension of the general theory of rigidity and stability undertaken by Connelly and Whiteley in [2]. The direct consequence of their work is the definition of crucial notions such as *first-* and *second order rigidity*, *prestress stability* and *rigidity* [2].

The hierarchical classification of the aforementioned notions with respect to the rigidity properties of frameworks gives us the necessary and sufficient condition to render a statically and kinematically indeterminate structure simply stiff. In summary, as shown in Figure 4, first-order rigidity implies prestress stability, which implies second-order rigidity, which at its turn implies global rigidity of structure. It is important to note that none of these implications are reversible. Thus **the crucial condition needed to be complied with is that a structure must be prestress stable in order to render it stiff**.

In concordance with Connelly and Whiteley's findings in [2] the prestress stable state of a pin-jointed structure can be checked by the **analysis of the tangent stiffness matrix** of the structure, which **must be positive definite**.

Fig. 4. *Classification of frameworks with respect to rigidity properties* [2]

6. Conclusions

The existence of the "ill-conditioned" frames which comply with Maxwell's original rule, although they are actually statically and kinematically indeterminate structures makes a compelling argument for the reformulation of the well-known rule in order to take into account all kinds of special configurations. As Calladine and Pellegrino showed in [1], [5] this can be achieved only by taking into account the number of the possible infinitesimal mechanisms and the independent states of self-stresses capable to stiffen them.

As the provided mechanism theory shows it is imperative that one can differentiate prestressable infinitesimal mechanism configurations from mechanisms of infinite order (finite mechanisms), which can not be rendered stiff by the means of pretensioning.

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