A NOTE ON FUNDAMENTAL GROUP LATTICES

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Abstract

The main goal of this note is to provide a new proof of a classical result about
projectivities between finite abelian groups. It is based on the concept of fundamental
group lattice, studied in our previous papers [8] and [9]. A generalization of this result
is also given.

2000 Mathematics Subject Classification: 20K01, 20D30

Key words: finite abelian groups, subgroup lattices, fundamental group lattices,
lattice isomorphisms.

1 Introduction

The relation between the structure of a group and the structure of its subgroup lattice
constitutes an important domain of research in group theory. One of the most interesting
problems concerning it is to study whether a group $G$ is determined by the subgroup
lattice of the $n$-th direct power $G^n$, $n \in \mathbb{N}^*$. In other words, if the $n$-th direct powers of
two groups have isomorphic subgroup lattices, are these groups isomorphic? For $n = 1$ it
is well-known that this problem has a negative answer (see [4]). The same thing can be
also said for $n = 2$, except for some particular classes of groups, as simple groups (see [5]),
finite abelian groups (see [3]) or abelian groups with the square root property (see [2]). In
the general case (when $n \geq 2$ is arbitrary) we recall Remark 1 of [2], which states that an
abelian group is determined by the subgroup lattice of its $n$-th direct power if and only if
it has the $n$-th root property. This follows from some classical results of Baer [1].

The starting point of our discussion is given by papers [8] and [9] (see also Section I.2.1
of [7]), where the concept of fundamental group lattice is introduced and studied. It gives
an arithmetic description of the subgroup lattice of a finite abelian group and has many
applications. Fundamental group lattices were successfully used in [8] to solve the problem
of existence and uniqueness of a finite abelian group whose subgroup lattice is isomorphic
to a fixed lattice and in [9] to count some types of subgroups of a finite abelian group. In
this paper they will be used to prove that the finite abelian groups are determined by the
subgroup lattices of their direct $n$-powers, for any $n \geq 2$. Notice that our proof is more
simple than the original one. A more general result will be also inferred.

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Most of our notation is standard and will usually not be repeated here. Basic definitions and results on groups can be found in [6]. For subgroup lattice notions we refer the reader to [4] and [7].

In the following we recall the concept of fundamental group lattice and two related theorems. Let $G$ be a finite abelian group and $L(G)$ be the subgroup lattice of $G$. Then, by the fundamental theorem of finitely generated abelian groups, exist (uniquely determined by $G$) numbers $k \in \mathbb{N}^*$ and $d_1, d_2, \ldots, d_k \in \mathbb{N} \setminus \{0, 1\}$ satisfying $d_1 \mid d_2 \mid \ldots \mid d_k$ such that

\[ G \cong \bigtimes_{i=1}^{k} \mathbb{Z}_{d_i}. \]

This decomposition of a $G$ into a direct product of cyclic groups together with the form of subgroups of $\mathbb{Z}^k$ (see Lemma 2.1 of [8]) leads us to the following construction:

Let $k \geq 1$ be an integer. Then, for every $(d_1, d_2, \ldots, d_k) \in (\mathbb{N} \setminus \{0, 1\})^k$, we consider the set $L(k; d_1, d_2, \ldots, d_k)$ consisting of all matrices $A = (a_{ij}) \in M_k(\mathbb{Z})$ which satisfy the conditions:

I. $a_{ij} = 0$, for any $i > j$,
II. $0 \leq a_{1j}, a_{2j}, \ldots, a_{j-1j} < a_{jj}$, for any $j = 1, k$,
III. $1) \ a_{11} \mid d_1,$
   \[ 2) \ a_{22} \mid \left(d_2, d_1 \frac{a_{12}}{a_{11}}\right), \]
   \[ 3) \ a_{33} \mid \left(d_3, d_2 \frac{a_{23}}{a_{22}}, d_1 \frac{a_{12}}{a_{22}}, \frac{a_{13}}{a_{11}}, \frac{a_{12}}{a_{22}}\right), \]
   \[ \vdots \]
   \[ k) \ a_{kk} \mid \left(d_k, d_{k-1} \frac{a_{k-1k}}{a_{k-1k-1}}, d_{k-2} \frac{a_{k-2k-1}}{a_{k-2k}}, \frac{a_{k-2k}}{a_{k-2k-1}}, \frac{a_{k-2k}}{a_{k-2k-1}a_{k-2k-2}}, \ldots, \right. \]
   \[ \left. \frac{a_{k-2k}}{a_{k-2k-1}a_{k-2k-2}a_{k-2k}}, \ldots, \frac{a_{11}}{a_{k-1k-1}a_{k-2k-1}a_{k-2k}}, \ldots, \frac{a_{11}}{a_{k-1k-1}a_{k-2k-1}a_{k-2k}}, \ldots, a_{11} \right), \]

where by $(x_1, x_2, \ldots, x_m)$ we denote the greatest common divisor of numbers $x_1, x_2, \ldots, x_m \in \mathbb{Z}$. On the set $L(k; d_1, d_2, \ldots, d_k)$ we introduce the ordering relation "$\leq$", defined as follows: for $A = (a_{ij}), B = (b_{ij}) \in L(k; d_1, d_2, \ldots, d_k)$, put $A \leq B$ if and only if we have

1)$ \ b_{11} \mid a_{11},$
2)$ b_{22} \left(\begin{array}{c} a_{11} a_{12} \\ b_{11} b_{12} \end{array}\right),$
3)$ \vdots \]
4)$ \frac{a_{11}}{a_{k-1k-1}a_{k-2k-1}a_{k-2k}}, \ldots, \frac{a_{11}}{a_{k-1k-1}a_{k-2k-1}a_{k-2k}}, \ldots, a_{11} \right), \]
shall write

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Then every integer

d
Theorem A. Let for

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In order to study when two fundamental group lattices are isomorphic (that is, when

\( \pi \) has

whenever the next three conditions are satisfied:

3)’ \( b_{33} | (a_{33}, \frac{a_{22} a_{23}}{b_{22}}, \frac{a_{11} a_{12} a_{13}}{b_{22} b_{11}}) \),

\[
\begin{pmatrix}
   a_{k-1} & a_{k-1} & a_{k-1} \\
   b_{k-1} & b_{k-1} & b_{k-1} \\
   b_{k-1} & b_{k-1} & b_{k-1}
\end{pmatrix}, \ldots,
\]

\[
\begin{array}{cccc}
   a_{k-1} & a_{k-1} & a_{k-1} \\
   b_{k-1} & b_{k-1} & b_{k-1} \\
   b_{k-1} & b_{k-1} & b_{k-1}
\end{array}, \ldots,
\]

Then \( L_{(k,d_1,d_2,\ldots,d_k)} \) forms a complete modular lattice with respect to \( \leq \), called a fundamental group lattice of degree \( k \). A powerful connection between this lattice and \( L(G) \) has been established in [8].

**Theorem A.** If \( G \) is a finite abelian group with the decomposition (\( * \)), then its subgroup lattice \( L(G) \) is isomorphic to the fundamental group lattice \( L_{(k,d_1,d_2,\ldots,d_k)} \).

In order to study when two fundamental group lattices are isomorphic (that is, when two finite abelian groups are lattice-isomorphic), the following notation is useful. For every integer \( n \geq 2 \), we denote by \( \pi(n) \) the set consisting of all primes dividing \( n \). Let

\( d_i, d_{i'} \in \mathbb{N} \setminus \{0, 1\}, i = \overline{1,k}, i' = \overline{1,k'} \), such that \( d_i | d_2 | \ldots | d_k \) and \( d_{i'} | d_2 | \ldots | d_k \). Then we shall write

\( (d_1, d_2, \ldots, d_k) \sim (d_{i_1}, d_{i_2}, \ldots, d_{i_{k'}}) \)

whenever the next three conditions are satisfied:

a) \( k = k' \).

b) \( d_i = d_{i'}, i = \overline{1,k-1} \).

c) The sets \( \pi(d_k) \setminus \pi \left( \prod_{i=1}^{k-1} d_i \right) \) and \( \pi(d_{i'}) \setminus \pi \left( \prod_{i=1}^{k-1} d_{i'} \right) \) have the same number of elements, say \( r \). Moreover, for \( r = 0 \) we have \( d_k = d_{i'} \) and for \( r \geq 1 \), by denoting \( \pi(d_k) \setminus \pi \left( \prod_{i=1}^{k-1} d_i \right) = \{p_1, p_2, \ldots, p_r\} \), \( \pi(d_{i'}) \setminus \pi \left( \prod_{i=1}^{k-1} d_{i'} \right) = \{q_1, q_2, \ldots, q_r\} \), we have

\[
\frac{d_k}{d_{i'}} = \prod_{j=1}^{r} \left( \frac{p_j}{q_j} \right)^{s_j},
\]

where \( s_j \in \mathbb{N}^* \), \( j = \overline{1,r} \).
The following theorem of [8] will play an essential role in proving our main results.

**Theorem B.** Two fundamental group lattices \( L_{(k;d_1,d_2,...,d_k)} \) and \( L_{(k';d'_1,d'_2,...,d'_{k'})} \) are isomorphic if and only if \((d_1,d_2,...,d_k) \sim (d'_1,d'_2,...,d'_{k'})\).

## 2 Main results

As we have already mentioned, large classes of non-isomorphic finite abelian groups exist whose lattices of subgroups are isomorphic. Simple examples of such groups are easily obtained by using Theorem B:

1. \( G = \mathbb{Z}_6 \) and \( H = \mathbb{Z}_{10} \) (cyclic groups),
2. \( G = \mathbb{Z}_2 \times \mathbb{Z}_6 \) and \( H = \mathbb{Z}_2 \times \mathbb{Z}_{10} \) (non-cyclic groups).

Moreover, Theorem B allows us to find a subclass of finite abelian groups which are determined by their lattices of subgroups (see also Proposition 2.8 of [8]).

**Theorem 2.1.** Let \( G \) and \( H \) be two finite abelian groups such that one of them possesses a decomposition of type (*) with \( \pi(d_k) = \pi \left( \prod_{i=1}^{k-1} d_i \right) \). Then \( G \cong H \) if and only if \( L(G) \cong L(H) \).

Next we shall focus on isomorphisms between the subgroup lattices of the direct \( n \)-powers of two finite abelian groups, for \( n \geq 2 \). An alternative proof of the following well-known result can be also inferred from Theorem B.

**Theorem 2.2.** Let \( G \) and \( H \) be two finite abelian groups. Then \( G \cong H \) if and only if \( L(G^n) \cong L(H^n) \) for some integer \( n \geq 2 \).

**Proof.** Let \( G \cong \prod_{i=1}^{k} \mathbb{Z}_{d_i} \) and \( H \cong \prod_{i=1}^{k'} \mathbb{Z}_{d'_i} \) be the corresponding decompositions (*) of \( G \) and \( H \), respectively, and assume that \( L(G^n) \cong L(H^n) \) for some integer \( n \geq 2 \). Then the fundamental group lattices

\[
L_{(k; d_1, d_1, ..., d_1, d_k, ..., d_k)}^{\text{n factors}} \quad \text{and} \quad L_{(k'; d'_1, d'_1, ..., d'_1, d'_k, ..., d'_k)}^{\text{n factors}}
\]

are isomorphic. By Theorem B, one obtains

\[
(d_1, d_1, ..., d_1, d_k, ..., d_k) \sim (d'_1, d'_1, ..., d'_1, d'_k, ..., d'_k)
\]

and therefore \( k = k' \) and \( d_i = d'_i \), for all \( i = 1, k \). These equalities show that \( G \cong H \), which completes the proof.
Clearly, two finite abelian groups $G$ and $H$ satisfying $L(G^m) \cong L(H^n)$ for some (possibly different) integers $m, n \geq 2$ are not necessarily isomorphic. Nevertheless, a lot of conditions of this type can lead to $G \cong H$, as the following theorem shows.

**Theorem 2.3.** Let $G$ and $H$ be two finite abelian groups. Then $G \cong H$ if and only if there are the integers $r \geq 1$ and $m_1, m_2, ..., m_r, n_1, n_2, ..., n_r \geq 2$ such that $(m_1, m_2, ..., m_r) = (n_1, n_2, ..., n_r)$ and $L(G^{m_i}) \cong L(H^{n_i})$, for all $i = 1, r$.

**Proof.** Suppose that $G$ and $H$ have the decompositions in the proof of Theorem 2. For every $i = 1, 2, ..., r$, the lattice isomorphism $L(G^{m_i}) \cong L(H^{n_i})$ implies that $km_i = k'n_i$, in view of Theorem B. Set $d = (m_1, m_2, ..., m_r)$. Then $d = \sum_{i=1}^{r} \alpha_i m_i$ for some integers $\alpha_1, \alpha_2, ..., \alpha_r$, which leads to

$$kd = k \sum_{i=1}^{r} \alpha_i m_i = \sum_{i=1}^{r} \alpha_i km_i = \sum_{i=1}^{r} \alpha_i k'n_i = k' \sum_{i=1}^{r} \alpha_i n_i.$$  

Since $d \mid n_i$, for all $i = 1, r$, we infer that $k' \mid k$. In a similar manner one obtains $k \mid k'$, and thus $k = k'$. Hence $m_i = n_i$ and the group isomorphism $G \cong H$ is obtained from Theorem 2.2.

Finally, we indicate an open problem concerning the above results.

**Open problem.** In Theorem 3 replace condition $(m_1, m_2, ..., m_r) = (n_1, n_2, ..., n_r)$ with other connections between numbers $m_i$ and $n_i$, $i = 1, 2, ..., r$, such that the respective equivalence be also true.

**References**


