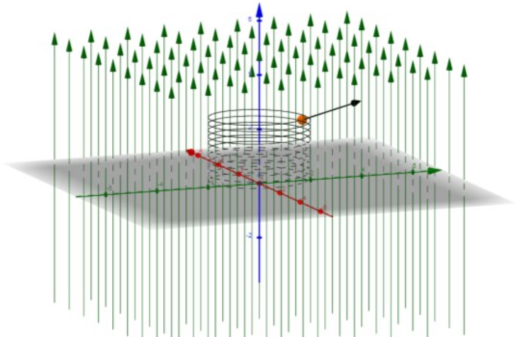
Electromagnetic field theory

Lecture notes and applications



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Chapter I. Electrostatics

Useful web resourses:

https://phet.colorado.edu/en/simulation/balloons-and-static-electricity
https://phet.colorado.edu/en/simulation/charges-and-fields
https://phet.colorado.edu/en/simulation/coulombs-law
https://phet.colorado.edu/en/simulation/capacitor-lab-basics
https://academo.org/demos/electric-field-line-simulator/

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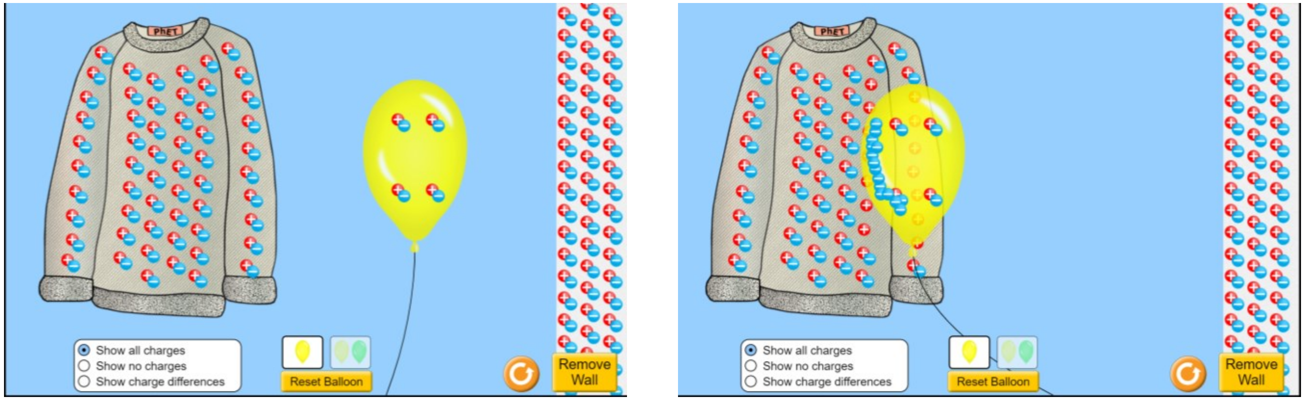
1.1 Introduction

Electrostatics is a branch of physics that studies electric charges at rest.

The first observation of an electric effect was made in 600 B.C. by a Greek mathematician and astronomer, Thales of Miletus. He discovered that a piece of amber, having been rubbed, would attract small bits of straw and feathers.
The Greek word for amber, ήλεκτρον, or electron, was thus the source of the word 'electricity'.

An ant inside Baltic amber



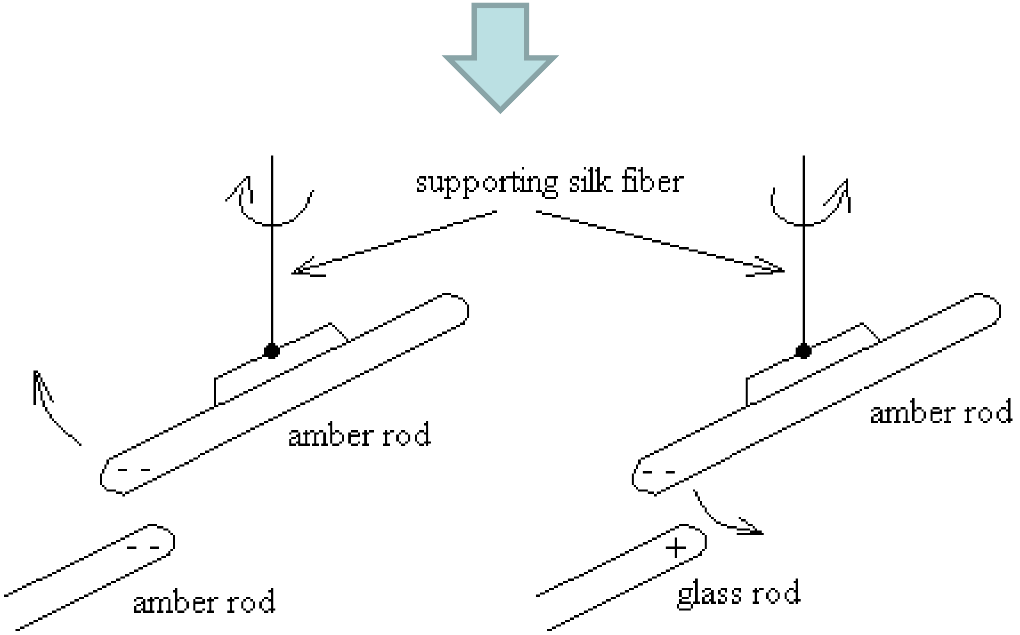
A plastic balloon having been rubbed on a sweeter would be attracted to it: https://phet.colorado.edu/en/simulation/balloons-and-static-electricity

Charles François du Fay (1698-1739): ***two types of electrification*** *are possible in:*

- *vitreous* materials - glass and mica
- *resinous* materials - amber, sulphur, hard rubber and resins

When rubbed with silk or cotton:

- the vitreous and resinous materials become oppositely electrified in the sense that bodies with *opposite electrification attract each other*
- bodies with the same electrification repel each other



1.2. Microscopic charge carriers

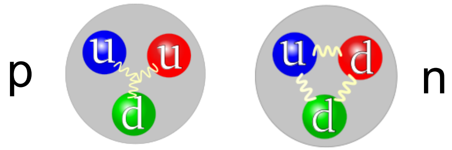
The measurement unit for the amount of charge: C (Coulomb)

By microscopic charge we understand charged particles and ions which can carry both positive and negative charges.
The numerical value of a charge can only be an ***integral multiple*** **of the elementary charge**, $|e|=1.6021892(46)\cdot 10^{-19}$ C.

Electron - is the material carrier of an elementary negative charge $e=-1.6021892(46)\cdot 10^{-19}$ C. It is usually assumed that an electron is a structureless point particle, i.e. the entire charge of an electron is concentrated at a point.
Contradictions: The energy of the electric field created by a point charge is infinite and hence the inertial mass of the point charge must also be infinite. This is in contradiction with the experiment since the electron rest mass is $m_e=9.11\cdot 10^{-31}$ kg.

Proton - is the carrier of a positive elementary charge. Unlike an electron, a proton is not considered as a point particle. Collisions (at very high energies) between electrons and protons give us conclusion *about charge distribution inside the proton*. A proton consists of two point *quarks* with a charge $+(2/3)\cdot|e|$ and one point quark with a charge $-(1/3)\cdot|e|$.
$m_p=1.672621777(74)\times 10^{-27}$ kg, $q=|e|=1.6021892(46)\cdot 10^{-19}$ C

Neutron - is a *neutral particle* which consists of two quarks with a charge $-|e|/3$ and one quark with a charge $+(2/3)\cdot|e|$.
$m_n=1.674927351(74)\times 10^{-27}$ kg, $q=0$.



Electrostatic generators

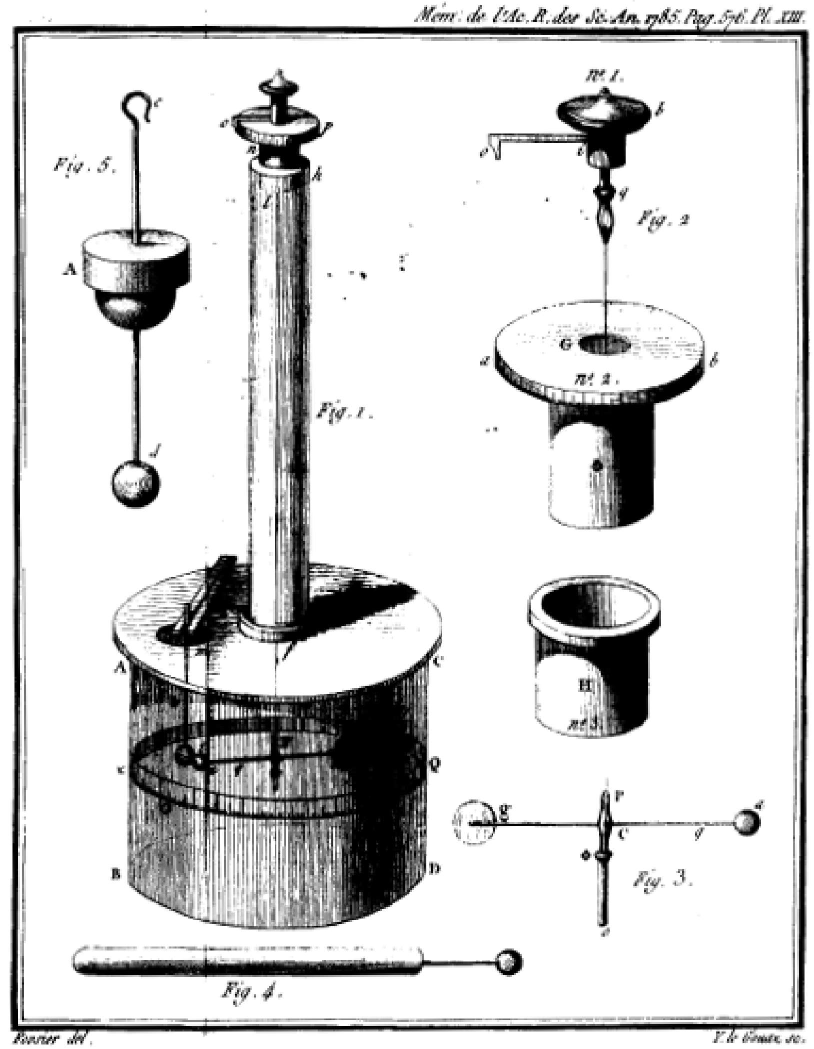
An electrostatic generator, or electrostatic machine, is an electromechanical generator that produces static electricity, or electricity at high voltage and low continuous current.

- In 18th century they became fundamental instruments in the studies about the science of electricity. Electrostatic generators operate by using manual (or other) power to transform mechanical work into electric energy.
- Electrostatic generators develop electrostatic charges of opposite signs rendered to two conductors, using only electric forces, and work by using moving plates, drums, or belts to carry electric charge to a high potential electrode.

The charge is generated by one of two methods: either the triboelectric effect (friction) or electrostatic induction.
The surface charge imbalance, which yields static electricity, can be generated by touching two differing surfaces together and then separating them due to the phenomena of contact electrification and the triboelectric effect. Rubbing two nonconductive objects generates a great amount of static electricity.

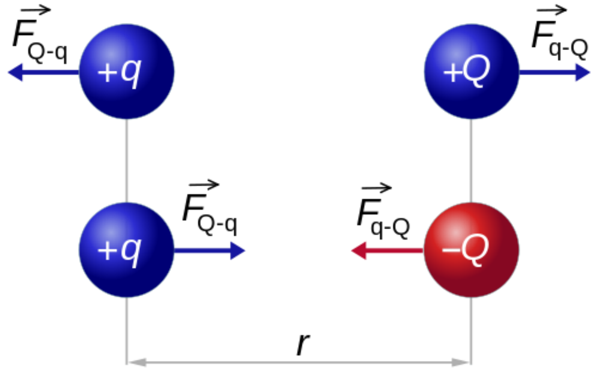
A Van de Graaff generator

1.3. The Coulomb's Law (1784)



Coulomb's torsion balance used to quantify the interaction force between two charged small spheres

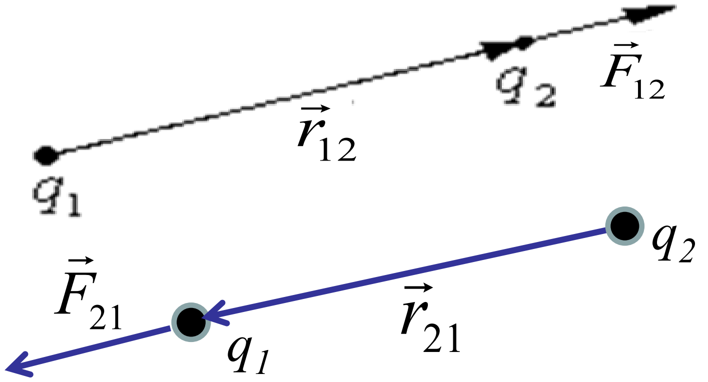
This is an experimental law of physics that quantifies the amount of force between two <u>stationary, electrically charged particles</u>. The electric force between charged bodies <u>at rest</u> is conventionally called <u>electrostatic force</u> or **<u>Coulomb force:</u>**



$$\left|\vec{F}_{Q-q}\right| = \left|\vec{F}_{q-Q}\right| = k\,\frac{|q\cdot Q|}{r^2}$$

If the two charges have the same sign, the electrostatic force between them is repulsive; if they have different sign, the force between them is attractive.

The force exerted by charge q_1 on charge q_2 is:



$$\boxed{F_{12} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}^2}} \text{ [N]} \qquad \boxed{\vec{F}_{12} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}^2}\frac{\vec{r}_{12}}{r_{12}}} \text{ [N]}$$

- scalar writing

$\vec{r}_{12} / r_{12}$ - unit vector

Cavendish made first experimental verification in 1722 but <u>did not published</u> the results

- one can write:

$$\vec{F}_{12} = \left(\frac{1}{4\pi\varepsilon_0}\frac{q_1}{r_{12}^2}\frac{\vec{r}_{12}}{r_{12}}\right)q_2 \quad \text{and} \quad \vec{F}_{21} = \left(\frac{1}{4\pi\varepsilon_0}\frac{q_2}{r_{21}^2}\frac{\vec{r}_{21}}{r_{21}}\right)q_1$$

- with: $\vec{F}_{12} + \vec{F}_{21} = 0$
- from Newton's 3rd Law

The **<u>charges</u>** q_1 and q_2 <u>create</u> in the space surrounding them **<u>a field</u>** named ***electric field*** which is characterized by the <u>strength</u> ***E*** which is a **<u>vector</u>**.

The *<u>field strength</u>*, $\vec{E}$, is a <u>local concept</u> - it *has a <u>definite value</u> at each point in space*.

$$\boxed{\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\frac{\vec{r}}{r} \quad \text{[N/C]=[V/m]}} \qquad \boxed{\vec{F} = q' \cdot \vec{E}}$$

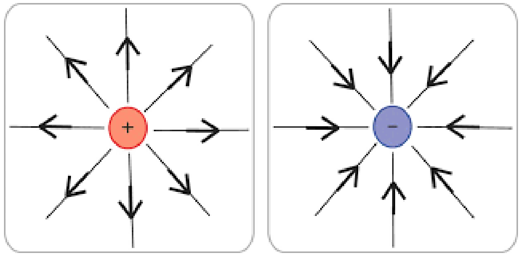
- the force that acts on a charge *q'* *placed in electric field*

$\varepsilon_0 = 8.854187817.. \times 10^{-12}$ F/m is the vacuum permittivity

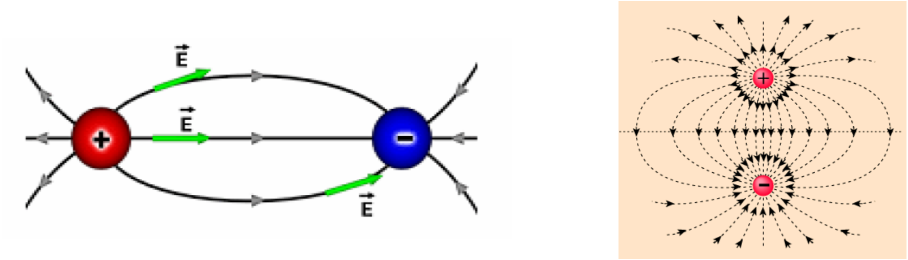
$\vec{r}$ - the position vector drawn from the point of location of the charge to the point where the field strength is measured

The electric field strength is a vector which, at each point of space, points directly away from q, if q is positive, and directly toward q, if q is negative

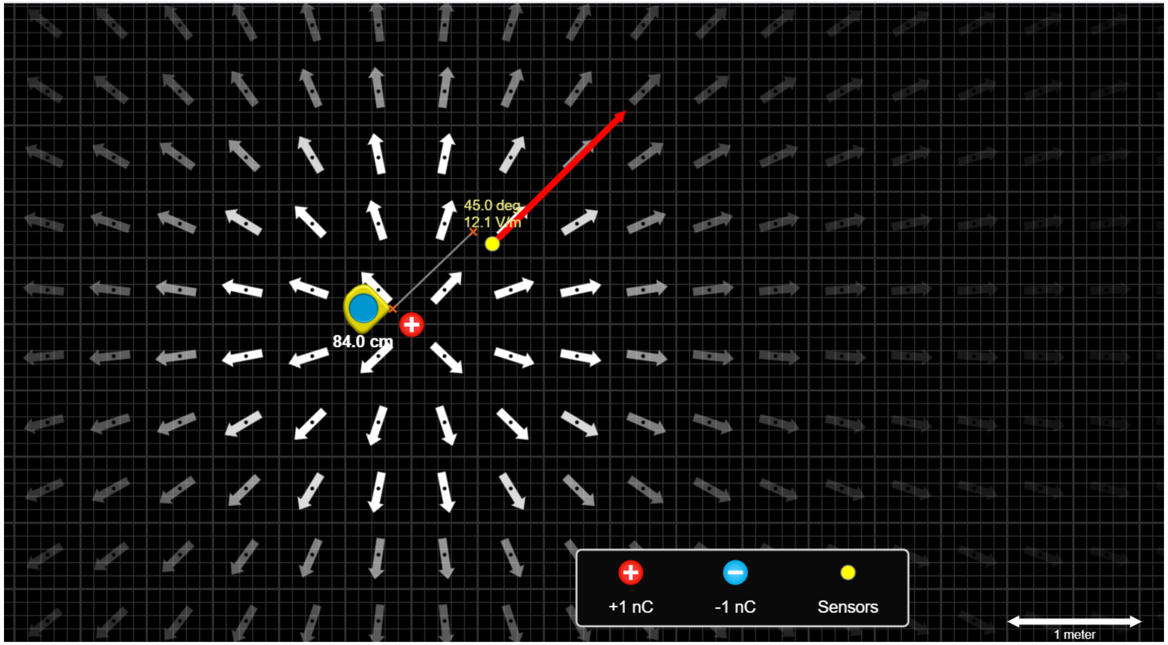
Lines of force for point charges:



for isolated charges



for a pair of positive and negative charges



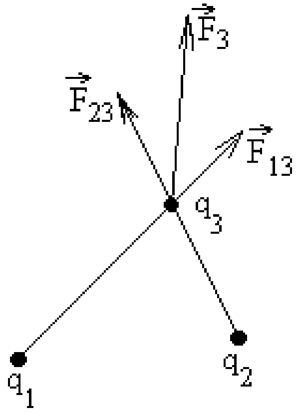
Simulation of electric field lines created by a 1 nC point charge, placed in vacuum; at 0.84 m, E=12.1 V/m

Check this value by using the expression of E.

https://phet.colorado.edu/sims/html/charges-and-fields/latest/charges-and-fields_en.html

1.3.1. Superposition Principle

Superposition principle for interaction of point charges is used to calculate the resulting force and, hence, the field created by such distribution.



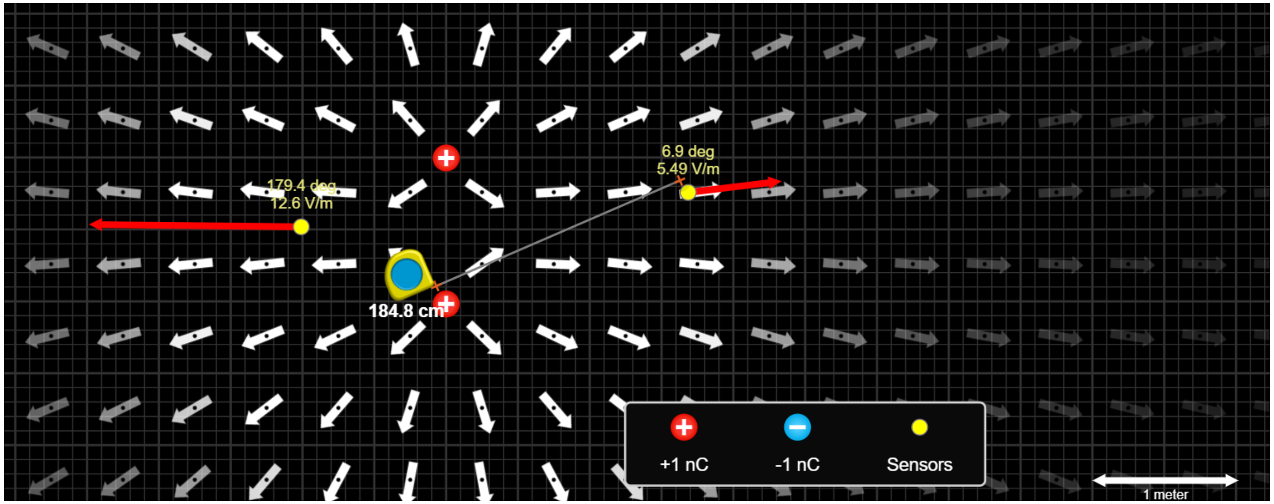
$$\vec{F}_3 = \vec{F}_{13} + \vec{F}_{23}$$

$$\vec{F}_{13} = q_3\vec{E}_{13}$$

$$\vec{F}_{23} = q_3\vec{E}_{23}$$

$$\vec{F}_3 = q_3\vec{E} \quad \Rightarrow \quad \vec{E} = \vec{E}_{13} + \vec{E}_{23}$$

- <u>generalization for an arbitrary number of charges</u>: $\vec{E} = \sum_i \vec{E}_i$



Simulation of the electric field created by 2 point charges of 1 nC each. https://phet.colorado.edu/en/simulation/charges-and-fields

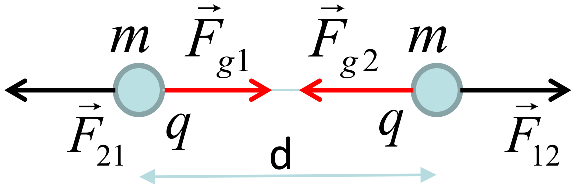
Test charge. The measurement of the electric field is reduced to the measurement of the force acting on a point charge. The point charge (yellow in this image) used for measuring the strength of an electric field is called ***<u>test charge</u>***

Ernshaw’s theorem: There exists no configuration of fixed charges, which would be stable in the absence of forces other than the forces of Coulomb’s interaction between the charges of the system. It is assumed that charges are held at various points in space by the force of no electrostatic origin

For example, the competition between Coulomb’s force and gravitational force can give a stable configuration of fixed charges

Solved application:

Two identical particles, carrying the same charge, *q*, and having the same mass, m, are placed in vacuum at a distance d between them. What should be the ratio between the charge, q, and the mass, m, for which the repelling Coulomb’s force cancels the gravitational force that acts between these particles?



$$F_{12} = F_{21} = \frac{1}{4\pi\varepsilon_0}\frac{qq}{d^2} = F_e$$

$$F_{g1} = F_{g2} = \gamma\frac{m\cdot m}{d^2} = F_g$$

$\gamma=6.67259\cdot10^{-11}$ $m^3kg^{-1}s^{-2}$ the *universal gravitational constant*

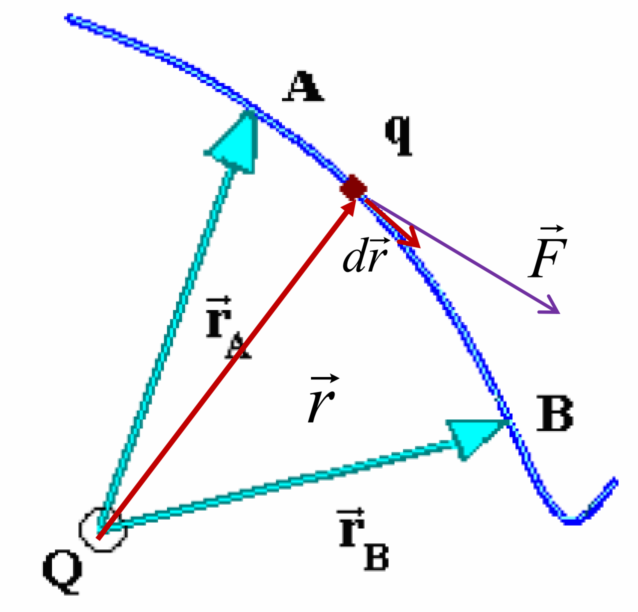
We find at equilibrium:

$$F_e = \frac{1}{4\pi\varepsilon_0}\frac{qq}{d^2} = \gamma\frac{m\cdot m}{d^2} = F_g \quad\Rightarrow\quad \frac{q^2}{m^2} = \gamma\cdot 4\pi\varepsilon_0$$

$$\Rightarrow \frac{q}{m} = \sqrt{\gamma\cdot 4\pi\varepsilon_0} = \sqrt{\frac{6.67259\cdot10^{-11}}{9\cdot10^{9}}} \simeq 8.61\cdot10^{-11}\ \ \text{C/kg}\quad \text{(specific charge)}$$

1.4. Potential nature of the electrostatic field

We can assume a fixed charge Q; in the field created by Q is moving a test charge q from A to B. The elementary work, dW, done to move the charge with d$\vec{r}$ can be expressed by:



$$dW = \vec{F}d\vec{r} = q\vec{E}\cdot d\vec{r} \qquad E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$$

The work done to move the charge q from A to B:

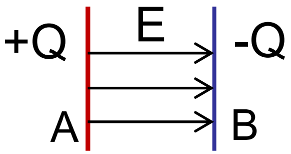
$$W_{AB} = q\int_{(A)}^{(B)}\vec{E}d\vec{r} = \frac{Qq}{4\pi\varepsilon_0}\int_{r_A}^{r_B}\frac{dr}{r^2}$$

$$W_{AB} = \frac{Qq}{4\pi\varepsilon_0}\left(-\frac{1}{r}\right)_{r_A}^{r_B} = \frac{Qq}{4\pi\varepsilon_0}\left(\frac{1}{r_A}-\frac{1}{r_B}\right)$$

$$\frac{W_{AB}}{q} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_A}-\frac{1}{r_B}\right) = \left(V_A - V_B\right) = U_{AB} \quad \left[\frac{J}{C}\right] = [V] \qquad V_A = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r_A}, \quad V_B = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r_B}$$

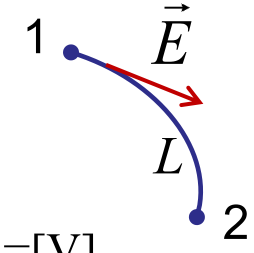
- where V_A and V_B are named **potentials** in A and B and U_{AB} is **potential difference**
- $U_{AB}=V_A-V_B$

In uniform electric field, E=ct., ➔ $U_{AB}=E\cdot d$



d – the distance between the metallic plates A and B

In a general way one can write the work performed in an electric field as:



$$dW = \vec{F}d\vec{l} = q\vec{E}\cdot d\vec{l}$$ - the elementary work

or $$\frac{dW}{q} = \frac{\vec{F}d\vec{l}}{q} = \vec{E}d\vec{l} \Rightarrow \frac{W}{q} = \int_{(1),L}^{(2)} \vec{E}d\vec{l} = U$$ [J/C]=[V]

- work performed per unit of charge

$$\boxed{W = q\cdot U}$$ with U potential difference (voltage)

The Potential nature of a Coulomb field

A force field is called a **potential field** if the work done upon a displacement in this field depends only on the ***initial and the final*** points of the path and does not depends on the trajectory. In mathematical language this can be expressed by:

$$\boxed{\oint_L \vec{E}d\vec{l} = 0}$$ and, if $\vec{E} = \sum \vec{E}_i$ then $$\boxed{\oint_L \vec{E}_i d\vec{l} = 0}$$

where L is an arbitrary closed contour over which the particle moves.

1.4.1. The electrostatic field is a potential field

Using the Stokes integral theorem, we find:

$$\oint_L \vec{E}d\vec{l} = \iint_S curl\vec{E}d\vec{S} = 0 \quad \text{for any } d\vec{S} \quad \Rightarrow \quad \boxed{curl\,\vec{E} \equiv rot\,\vec{E} = \nabla \times \vec{E} = 0}$$

which is ***the differential form*** *of the statement that* ***the electrostatic field is a potential field***

Here: $$curl\vec{E} = rot\,\vec{E} = \nabla \times \vec{E} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}, \quad \text{with } \nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

nabla operator

So, we can define a **scalar potential** φ ***(or simply V)*** from which $\vec{E}$ can be calculated as:

$$\boxed{\vec{E} = -grad\,V = -\nabla V = -\left(\vec{i}\frac{\partial V}{\partial x} + \vec{j}\frac{\partial V}{\partial y} + \vec{k}\frac{\partial V}{\partial z}\right)}$$

It comes that the potential, V, of a given electric field is defined only to within an additive constant.

- for a point charge q: $$\boxed{V(r) = -\frac{q}{4\pi\varepsilon_0}\int_r^{\infty}\frac{dr}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}, \quad \text{if } V(\infty) \to 0}$$

If: $$\vec{E} = \vec{E}_1 + \vec{E}_2 = -grad\, V_1 - grad\, V_2 = -grad(V_1 + V_2) = -grad\, V$$

with: $V = V_1 + V_2$

$$\boxed{V(x,y,z) = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{r_i}}$$ for a system of point charges

Field potential of continuously distributed charges

For a very large number of charges (e.g. comparable with the number of atoms or molecules inside the substance, the discrete approach of summation cannot be applied. In this case, can be used a continuous charge distribution approach even if, in reality, the charges are discrete.

In function of the system's extend in space, we can consider:

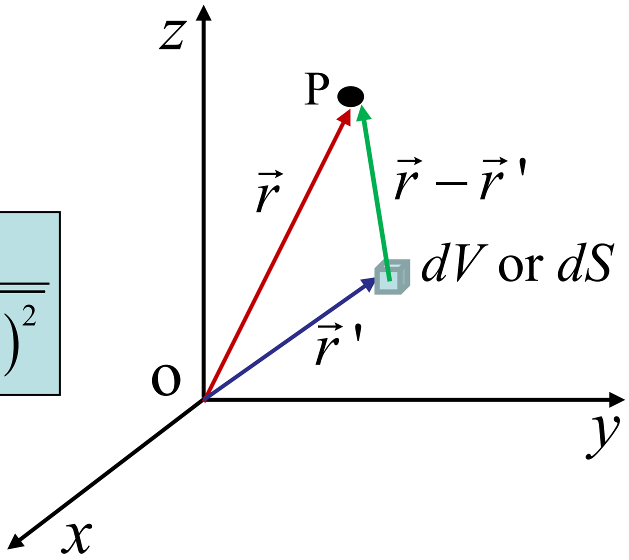
a) Volume charge density, ρ, is defined as $\rho = q/V$ [C/m^3] if the charge is uniformly distributed inside the substance or, more general, as $\rho = dq/dV$ [C/m^3]. This approach is used for objects with comparable extends over x, y and z axes;
b) Surface charge density, σ, is defined as $\sigma = q/S$ [C/m^2] if the charge is uniformly distributed inside the substance or, more general, as $\sigma = dq/ds$ [C/m^2]. This approach is used for objects with extends over x and y axes much larger than over z axis. In other words this is used to describe two dimensional systems like like sheet of paper, etc.;
c) Linear charge density, γ, is defined as $\gamma = q/l$ [C/m] if the charge is uniformly distributed inside the substance or, more general, as $\gamma = dq/dl$ [C/m]. This approach is used for objects with unidimensional objects, like strings, carbon nanotubes, etc.

The potential generated in point P by a continuous charge distribution can be expressed by:

$$V = \frac{1}{4\pi\varepsilon_0}\iiint_V \frac{\rho\cdot dV}{\left|\vec{r}-\vec{r}\,'\right|}$$

$$dV = dx'\cdot dy'\cdot dz'$$

$$V(x,y,z) = \frac{1}{4\pi\varepsilon_0}\iiint_V \frac{\rho(x',y',z')\cdot dx'dy'dz'}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$



When the surface charge distribution is considered the above formula becomes:

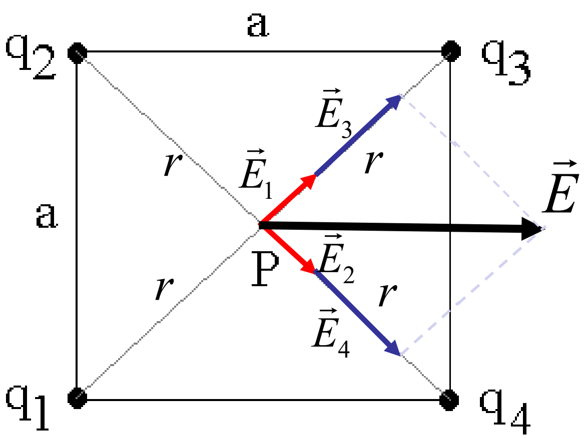
$$V = \frac{1}{4\pi\varepsilon_0}\iint_S \frac{\sigma dS}{\left|\vec{r}-\vec{r}\,'\right|}$$

$$dS = dx'\cdot dy'$$

$$V(x,y,z) = \frac{1}{4\pi\varepsilon_0}\iint_S \frac{\sigma(x',y',z')\cdot dx'dy'}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

1.5. Solved and proposed applications – part 1

1. Four point charges are distributed on the corners of a square of side a=1 m as in the figure below. The charges are $q_1 = q_2 = 1$ nC, $q_3 = q_4 = -2$ nC. Find a) the field strength and b) the potential at the point P lying at the intersection of the diagonals.



a) $$E_1 = E_2 = \frac{q_1}{4\pi\varepsilon_0}\frac{1}{r^2} = \frac{q_2}{4\pi\varepsilon_0}\frac{1}{r^2}$$

$$E_3 = E_4 = \frac{|q_3|}{4\pi\varepsilon_0}\frac{1}{r^2} = \frac{|q_4|}{4\pi\varepsilon_0}\frac{1}{r^2}\ ;\ d = \sqrt{2a^2} = a\sqrt{2} \Rightarrow r = a\frac{\sqrt{2}}{2}$$

$$E = \sqrt{E_{13}^2 + E_{24}^2} = \sqrt{\left(\frac{q_1}{4\pi\varepsilon_0}\frac{1}{r^2} + \frac{|q_3|}{4\pi\varepsilon_0}\frac{1}{r^2}\right)^2 + \left(\frac{q_2}{4\pi\varepsilon_0}\frac{1}{r^2} + \frac{|q_4|}{4\pi\varepsilon_0}\frac{1}{r^2}\right)^2}$$

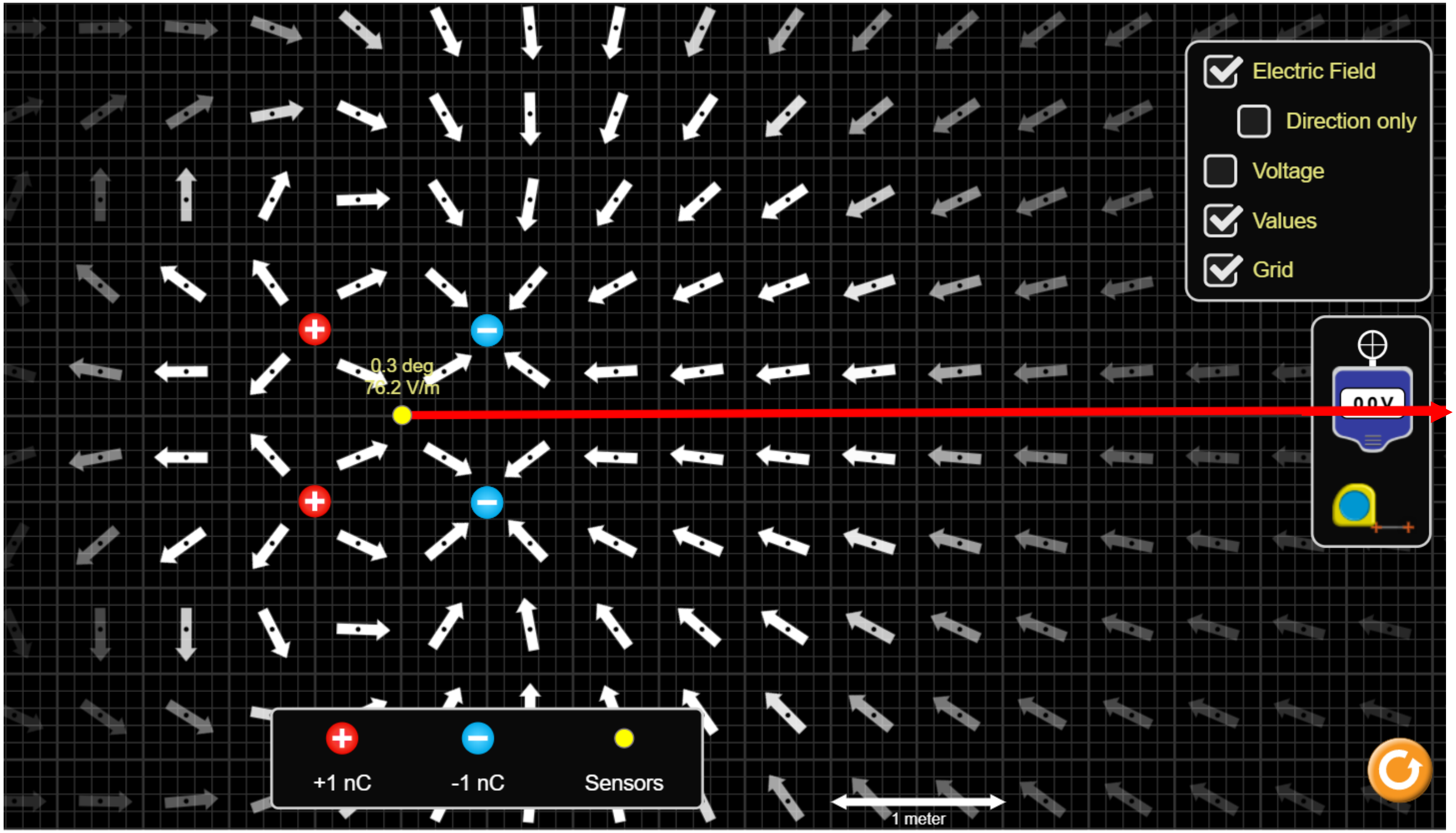
$$E = \sqrt{2\left(\frac{q_1}{4\pi\varepsilon_0}\frac{1}{r^2} + \frac{|q_3|}{4\pi\varepsilon_0}\frac{1}{r^2}\right)^2} = \left(\frac{q_1}{4\pi\varepsilon_0}\frac{1}{r^2} + \frac{|q_3|}{4\pi\varepsilon_0}\frac{1}{r^2}\right)\sqrt{2}$$

$$E = \frac{1}{4\pi\varepsilon_0}\cdot\frac{1}{r^2}\left(q_1 + |q_3|\right)\sqrt{2} \quad \text{with} \quad r^2 = \frac{a^2}{2}$$

$$E = \frac{1}{4\pi\varepsilon_0}\cdot\frac{2}{a^2}\left(q_1 + |q_3|\right)\sqrt{2} \approx 9\cdot10^9\,\frac{2}{1}(1+2)\cdot10^{-9}\sqrt{2}$$

$$E = 76.37\ \text{V/m}$$

Simulation results show a field of 76.2 V/m directed like in the figure below.
Red charges are of 1 nC whereas blue charges are of 2 nC; the distance between them is 1 m.
The simulator allows using charges which are multiples of ±1 nC



- ☑ Electric Field
  - ☐ Direction only
- ☐ Voltage
- ☑ Values
- ☑ Grid

https://phet.colorado.edu/sims/html/charges-and-fields/latest/charges-and-fields_en.html

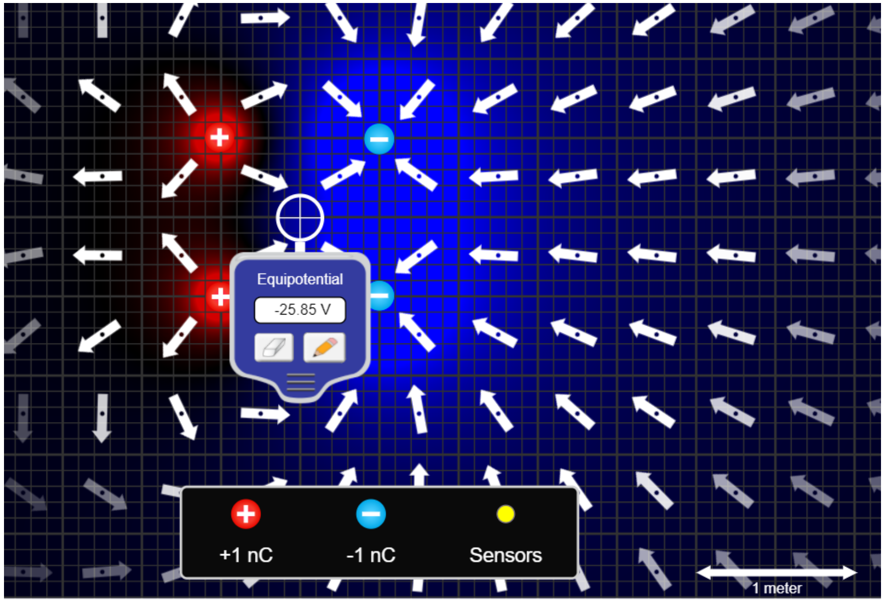
Cont.

b) the potential in point P

$$V = V_1 + V_2 + V_3 + V_4 = \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\left(q_1 + q_2 + q_3 + q_4\right) \qquad r = a\frac{\sqrt{2}}{2}$$

$$V \approx 9\cdot 10^9 \frac{2}{1\cdot\sqrt{2}}\left(1+1+(-2)+(-2)\right)\cdot 10^{-9} \quad [V]$$

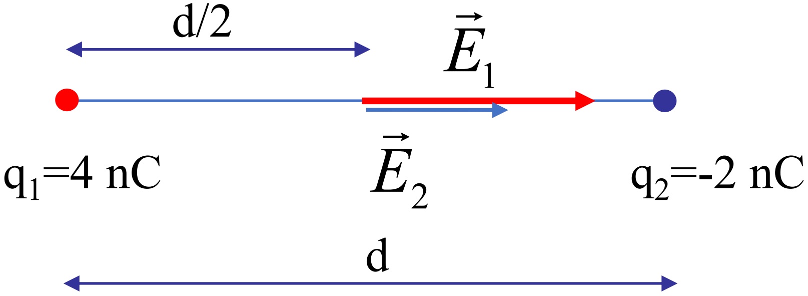
$$V = \frac{18}{\sqrt{2}}\cdot(-2) = -\frac{36}{\sqrt{2}} = -\frac{36\cdot\sqrt{2}}{2} = -18\sqrt{2} = \text{-}25.46 \quad [V]$$



Simulation results show a potential of -25.85 V. What can be the source of errors?

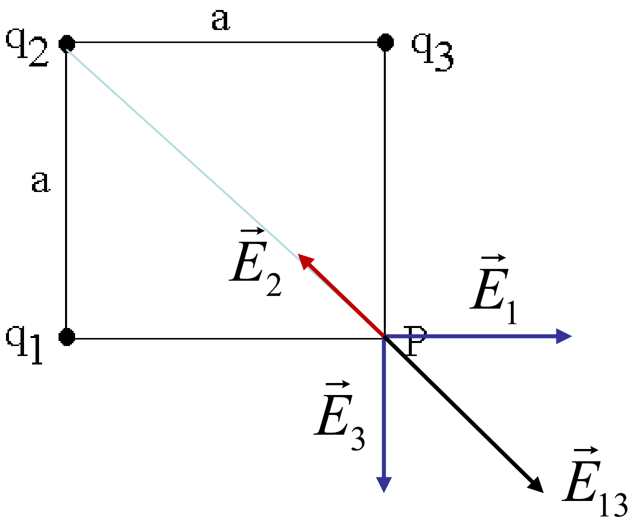
https://phet.colorado.edu/sims/html/charges-and-fields/latest/charges-and-fields_en.html

2. Two point charges q_1=4 nC and q_2=-2 nC are set apart at a distance d=1 m. Calculate the electric field strength at a point P lying at a distance d/2 between the charges, on the line uniting them.



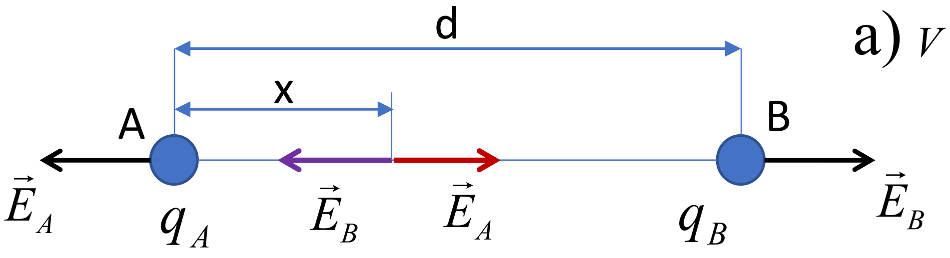
$$\vec{E} = \vec{E}_1 + \vec{E}_2 \;\Rightarrow\; E = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{\left(d/2\right)^2} + \frac{1}{4\pi\varepsilon_0}\frac{\left|q_2\right|}{\left(d/2\right)^2}$$

3. Three point charges are distributed on the corners of a square of side *a*=1 m, as in the figure below. The charges are $q_1=q_3$=1 nC and q_2=-2 nC. Find the field strength and the potential at the point P lying on the fourth corner of the square.



4. Two particles, A and B, carrying charges $q_A=2\ \mu C$ and $q_B=4\ \mu C$, are placed in air, at a distance d=1.8 m between them, see the figure bellow.

Calculate: (a) the electric field strength (the intensity of the electric field) in points on the line AB where the potential is zero and (b) the electric potential in points where the resulting electric field strength is zero.



a) $V(x)=V_A(x)+V_B(x)=\dfrac{1}{4\pi\varepsilon_0}\dfrac{q_A}{x}+\dfrac{1}{4\pi\varepsilon_0}\dfrac{q_B}{d-x}>0$

Because q_A and $q_B>0$ ➔ there are no points on the line AB for which V=0; in fact nowhere V=0.

b) $\vec{E}_A+\vec{E}_B=0 \Rightarrow E_A=E_B \Rightarrow \dfrac{q_A}{4\pi\varepsilon_0}\dfrac{1}{x^2}=\dfrac{q_B}{4\pi\varepsilon_0}\dfrac{1}{(d-x)^2} \Rightarrow \dfrac{q_A}{x^2}=\dfrac{q_B}{(d-x)^2}$

$$q_A(d-x)^2=q_Bx^2 \Rightarrow q_Ad^2-2q_Ad\cdot x+q_A\cdot x^2-q_B\cdot x^2=0$$

$$(q_A-q_B)\cdot x^2-2q_Ad\cdot x+q_Ad^2=0$$

$$(q_B-q_A)\cdot x^2+2q_Ad\cdot x-q_Ad^2=0 \Rightarrow x_{1,2}=\frac{-2q_Ad\pm\sqrt{4q_A^2d^2+4(q_B-q_A)q_Ad^2}}{2(q_B-q_A)}$$

$$x=\frac{-2q_Ad+\sqrt{4q_A^2d^2+4(q_B-q_A)q_Ad^2}}{2(q_B-q_A)}=\frac{-2q_Ad+\sqrt{4q_Aq_Bd^2}}{2(q_B-q_A)}=\frac{-2q_Ad+2d\sqrt{q_Aq_B}}{2(q_B-q_A)}=1,8\frac{-2+\sqrt{8}}{2}=0.745\text{ m}$$

- the negative solution has no physical meaning

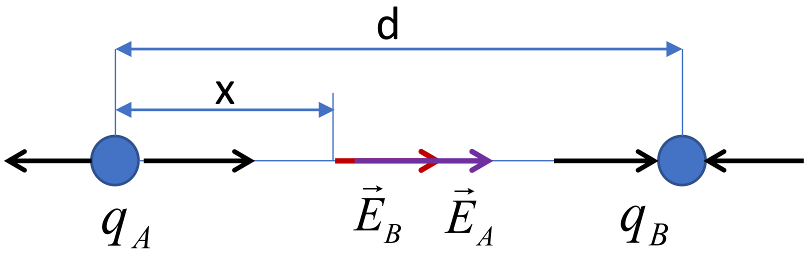
Cont.

$$V(x)=V_A(x)+V_B(x)=\frac{1}{4\pi\varepsilon_0}\frac{q_A}{x}+\frac{1}{4\pi\varepsilon_0}\frac{q_B}{d-x}=\frac{1}{4\pi\varepsilon_0}\left(\frac{q_A}{x}+\frac{q_B}{d-x}\right)$$

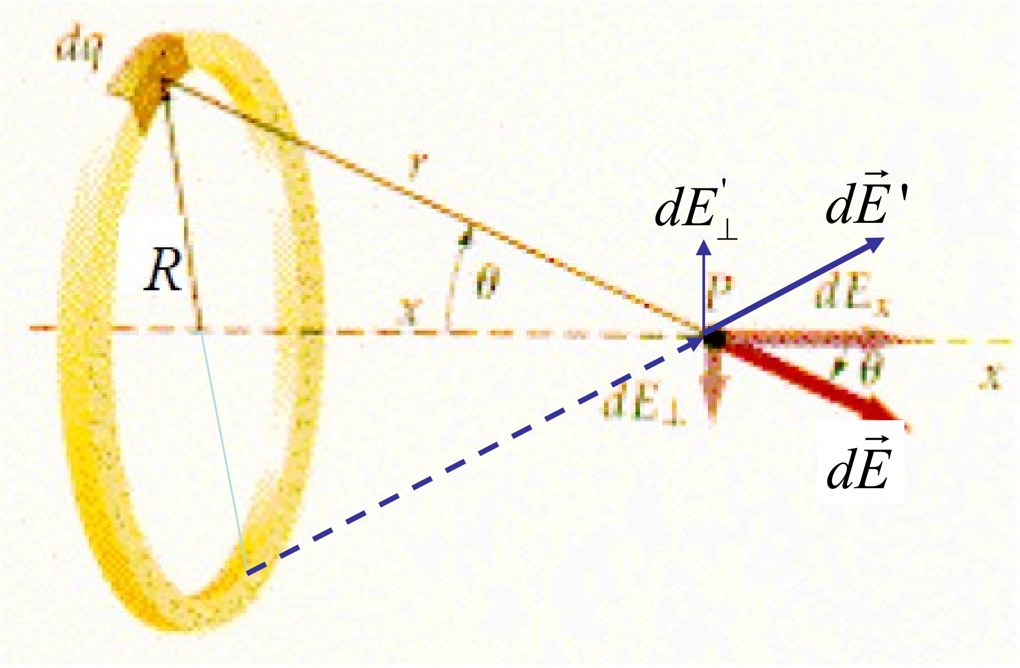
with x=0,745 m

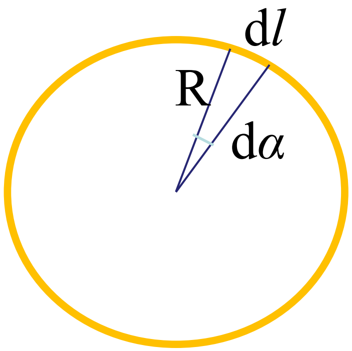
$$V(0,745\,m)=9\cdot 10^9\left(\frac{2}{0,745}+\frac{4}{1,8-0,745}\right)10^{-6}\simeq 5,83\cdot 10^4\ \text{V}$$

5. Two particles, A and B carrying charges q_A=2 µC and q_B=-4 µC, are placed in air, at a distance d=1.8 m between them, see the figure bellow.
Calculate: (a) the electric field strength (the intensity of the electric field) in points on the line AB where the potential is zero and (b) the electric potential in points where the resulting electric field strength is zero.



6. Calculate the electric field strength in a point situated at distance x from the center of a ring of radius *R*, uniformly charged with linear charge density γ.





$$dq = \gamma \cdot dl = \gamma \cdot R \cdot d\alpha$$

$$dE = \frac{dq}{4\pi\varepsilon_0}\frac{1}{r^2} \qquad r = \left(x^2 + R^2\right)^{1/2}$$

$d\vec{E}_{\perp} + d\vec{E}'_{\perp} = 0$ ⇨ it remains only the x component of the field

$$dE_x = dE\cos\theta = \frac{dq}{4\pi\varepsilon_0}\frac{1}{r^2}\cos\theta = \frac{dq}{4\pi\varepsilon_0}\frac{1}{r^2}\frac{x}{\left(x^2+R^2\right)^{1/2}}$$

$$dE_x = \frac{dq}{4\pi\varepsilon_0}\frac{1}{x^2+R^2}\frac{x}{\left(x^2+R^2\right)^{1/2}} = \frac{dq}{4\pi\varepsilon_0}\frac{x}{\left(x^2+R^2\right)^{3/2}} = \frac{\gamma R d\alpha}{4\pi\varepsilon_0}\frac{x}{\left(x^2+R^2\right)^{3/2}}$$

$$dq = \gamma dl = \gamma R d\alpha$$

⇩

$$E_x = \int_0^{2\pi} \frac{\gamma R}{4\pi\varepsilon_0}\frac{x}{\left(x^2+R^2\right)^{3/2}}\cdot d\alpha$$

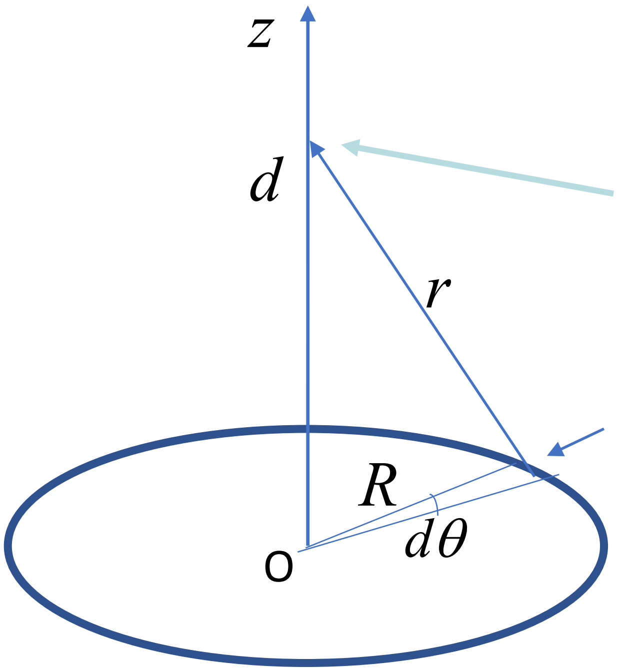
$$E = E_x = \frac{2\pi\cdot\gamma R}{4\pi\varepsilon_0}\frac{x}{\left(x^2+R^2\right)^{3/2}}$$

⇩

$$E = E_x = \frac{\gamma R x}{2\varepsilon_0\left(x^2+R^2\right)^{3/2}}$$

$E = 0$ if x=0 or x $\to \infty$ ⇨ The electric field is 0 in the centre of the ring

7. Calculate the electrical potential generated, by a circular ring with radius R, which carries a charge Q uniform distributed, in a point situated on the axis of symmetry at a distance d above the ring surface.



$$\gamma = \frac{Q}{2\pi R}$$ linear charge density

$$dV = \frac{dq}{4\pi\varepsilon_0}\frac{1}{r} \Rightarrow dV = \frac{\gamma dl}{4\pi\varepsilon_0}\frac{1}{\sqrt{d^2+R^2}} = \frac{\gamma R d\theta}{4\pi\varepsilon_0}\frac{1}{\sqrt{d^2+R^2}}$$

$dq = \gamma \cdot dl\,;\ \gamma = Q/2\pi R$ the charge is uniform distributed

$$dl = R \cdot d\theta$$

$$V = \int_0^{2\pi} \frac{\gamma R}{4\pi\varepsilon_0}\frac{1}{\sqrt{d^2+R^2}} \cdot d\theta = \frac{2\pi \cdot \gamma R}{4\pi\varepsilon_0}\frac{1}{\sqrt{d^2+R^2}} = \frac{\gamma R}{2\varepsilon_0}\frac{1}{\sqrt{d^2+R^2}}$$

$$V = \frac{Q \cdot R}{2\pi R \cdot 2\varepsilon_0}\frac{1}{\sqrt{d^2+R^2}} = \frac{Q}{4\pi \cdot \varepsilon_0}\frac{1}{\sqrt{d^2+R^2}} \sim \frac{1}{d}$$

if $d \gg R \quad \Rightarrow \quad \mathrm{V} = \frac{Q}{4\pi \cdot \varepsilon_0}\frac{1}{d\sqrt{1+R^2/d^2}} \approx \frac{Q}{4\pi \cdot \varepsilon_0}\frac{1}{d}$

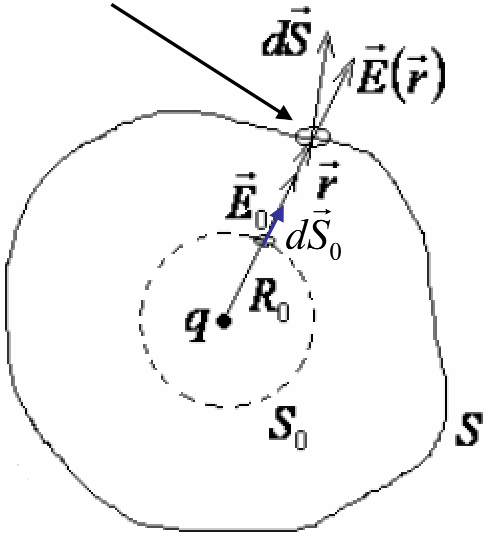
$$d = 0 \Rightarrow V = \frac{Q}{4\pi\varepsilon_0}\frac{1}{R}$$

- the potential behaves like for a point charge, Q, if d>>R

1.6. Gauss's law for electric field

a) Suppose that a point charge q is inside of a volume V bounded by a closed surface S

The flux of the electric field through an elementary surface dS is: $d\Phi = \vec{E} \cdot d\vec{S}$



- the flux of the field $\vec{E}$ through an arbitrary closed surface S

$$\Phi = \oiint_S \vec{E} d\vec{S} \qquad \text{with} \qquad \vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \frac{\vec{r}}{r}$$

- to estimate Φ we can calculate the flux of the field $\vec{E}$ through a spherical surface S_0

$$\Phi_0 = \oiint_{S_0} \vec{E}_0 d\vec{S}_0 = \frac{q}{4\pi\varepsilon_0 R_o^2} \oiint_{S_0} dS_0 = \frac{q}{\varepsilon_0} \qquad \vec{E}_0 \parallel d\vec{S}_0$$

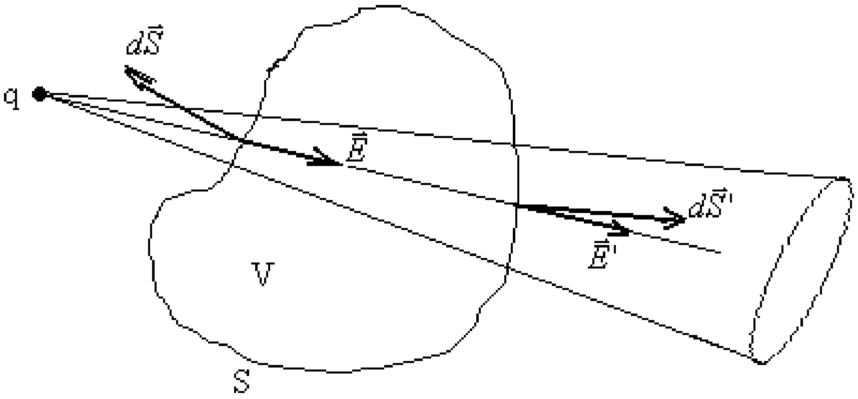
$$\text{with} \quad E_0 = \frac{1}{4\pi\varepsilon_0} \frac{q}{R_0^2} \quad \text{and} \quad \oiint_{S_0} dS_0 = 4\pi R_0^2$$

q is in the center of a spherical surface S_0

But $\Phi_0 = \Phi$ because the same number of electric field lines crosses S and S_0

⇒ $$\boxed{\Phi = \frac{q}{\varepsilon_0}}$$ **Gauss electrostatic theorem for a point charge**

b) The flux of $\vec{E}$ through a closed surface when a point charge is located outside the volume bounded by it



$$d\Phi = \vec{E}\cdot d\vec{S} \quad \text{and} \quad d\Phi' = \vec{E}'\cdot d\vec{S}'$$

$$d\Phi = -d\Phi' \quad \Rightarrow \quad d\Phi + d\Phi' = 0$$

$$\Downarrow$$

For a point charge located outside the volume V the flux of the field $\vec{E}$ through a closed surface is 0

$$\boxed{\Phi = 0}$$

c) A system of point charges:

- applying the principle of superposition $\vec{E} = \sum \vec{E}_i$

$$\Phi = \oiint_S \vec{E}d\vec{S} = \sum \oiint_S \vec{E}_i d\vec{S} \quad \Rightarrow \quad \boxed{\Phi = \oiint_S \vec{E}d\vec{S} = \frac{1}{\varepsilon_0}\sum_V q_i = \frac{1}{\varepsilon_0}Q} \qquad Q = \sum_V q_i$$

- for a continuous charge distribution with a volume charge density ρ: $\boxed{Q = \iiint_V \rho dV}$

$$\boxed{\Phi = \oint_S \vec{E}d\vec{S} = \frac{1}{\varepsilon_0}\sum_V q_i = \frac{1}{\varepsilon_0}\iiint_V \rho dV}$$

ρ=dQ/dV [C/m^3]

1.6.1. Differential form of Gauss's law. Maxwell's equation for $div\,\vec{E}$

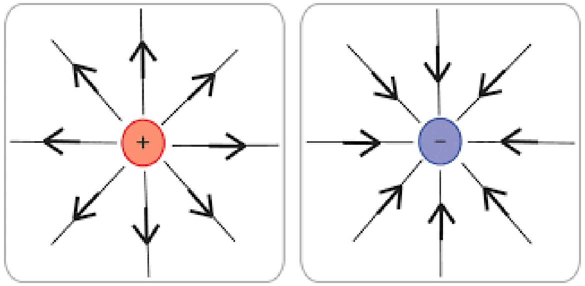
$\Phi = \oint\limits_S \vec{E}d\vec{S} = \int\limits_V div\vec{E}\,dV$ - obtained from Gauss' formula: $\oiint\limits_S \vec{f}d\vec{S} = \iiint\limits_V div\vec{f}\cdot dV$

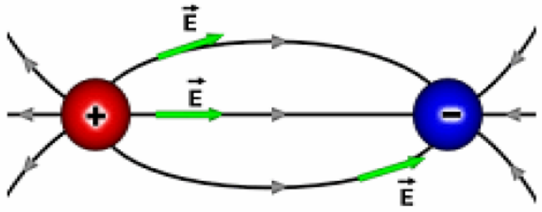
$$\boxed{div\vec{f} = \frac{df_x}{dx} + \frac{df_y}{dy} + \frac{df_z}{dz}}$$

$$\Phi = \frac{q}{\varepsilon_0} = \frac{1}{\varepsilon_0}\iiint \rho dV \Rightarrow \iiint\limits_V \left(div\,\vec{E} - \frac{\rho}{\varepsilon_0} \right) dV = 0$$

$$\boxed{div\,\vec{E} = \frac{\rho}{\varepsilon_0}}$$ - the differential form of Gauss's law

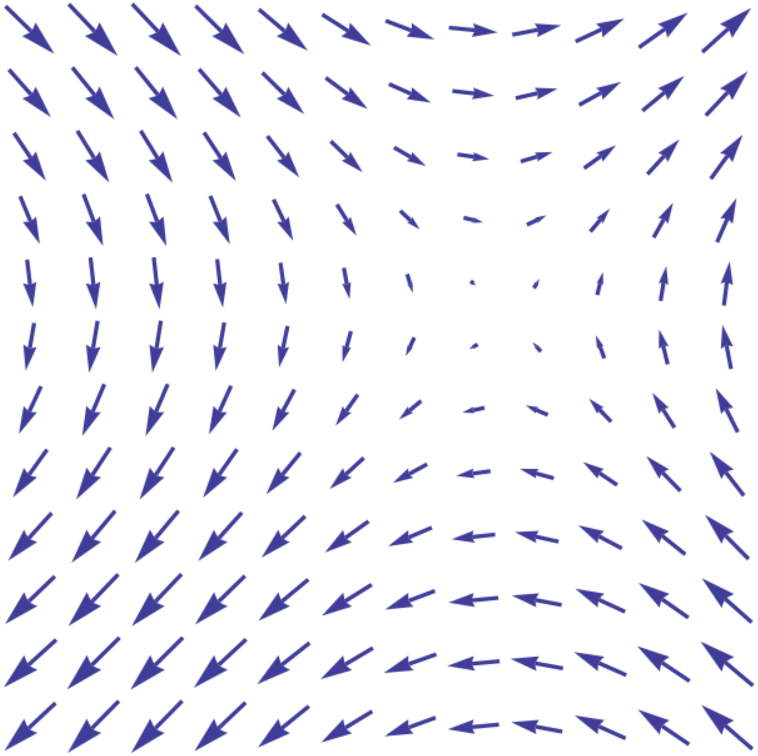
- the field lines start where $div\vec{E} > 0$ and terminate where $div < 0$;
- the field lines originate at positive charges and terminate at negative ones;
- this equation is also true for any arbitrary motion of charges.





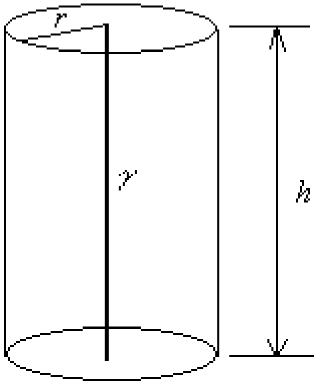
1.6.2. Vector field

- a **vector field** is an assignment of a vector to each point in space
- a vector field in the plane can be visualized as a collection of arrows with a given magnitude and direction each attached to a point in the plane.
- vector fields are often used to model, for example, the speed and direction of a moving fluid throughout space, or the strength and direction of some force, such as the magnetic or gravitational force, as it changes from point to point.



1.7. Solved and proposed applications – part 2

1. Find the field strength due to a very long charged filament with linear charge density γ.



Applying the Gauss theorem:

$$\oiint_S \vec{E}d\vec{S} = \frac{Q}{\varepsilon_0} \qquad Q = \gamma\, h$$

The flux $\vec{E}$ of through the bases is equal to 0

⇓

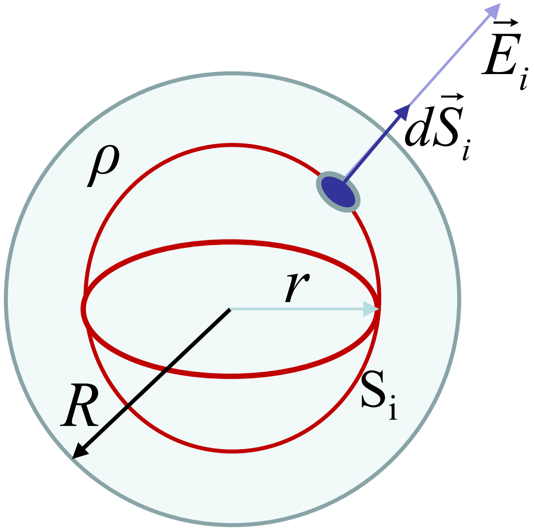
$$\oiint_S \vec{E}d\vec{S} = \iint_{S_{cylinder}} \vec{E}d\vec{S} = E \cdot 2\pi r r \cdot h$$

⇓

$$\boxed{E = \frac{1}{2\pi\varepsilon_0}\frac{\gamma}{r} \Rightarrow E \sim \frac{1}{r}}$$

2. A dielectric sphere of radius *R,* and dielectric permittivity ε_0, contains electric charge with uniform charge density ρ. Calculate the electric field as a function of distance *r* from the centre of the sphere, both inside and outside the sphere.

Hint. Use the Gauss electrostatic theorem



Because of the spherical symmetry, E has the same value on each point of the surface and $\vec{E}||d\vec{S}$

a) $r \le R$ ⇨ the field inside the sphere, E_i

$$\oiint_{S_i} \vec{E}_i d\vec{S}_i = \frac{Q_i}{\varepsilon_0} \Rightarrow E_i \oiint_{S_i} dS_i = \frac{Q_i}{\varepsilon_0}\ ; \qquad \rho = \frac{Q}{\frac{4\pi R^3}{3}} = \frac{3Q}{4\pi R^3}$$

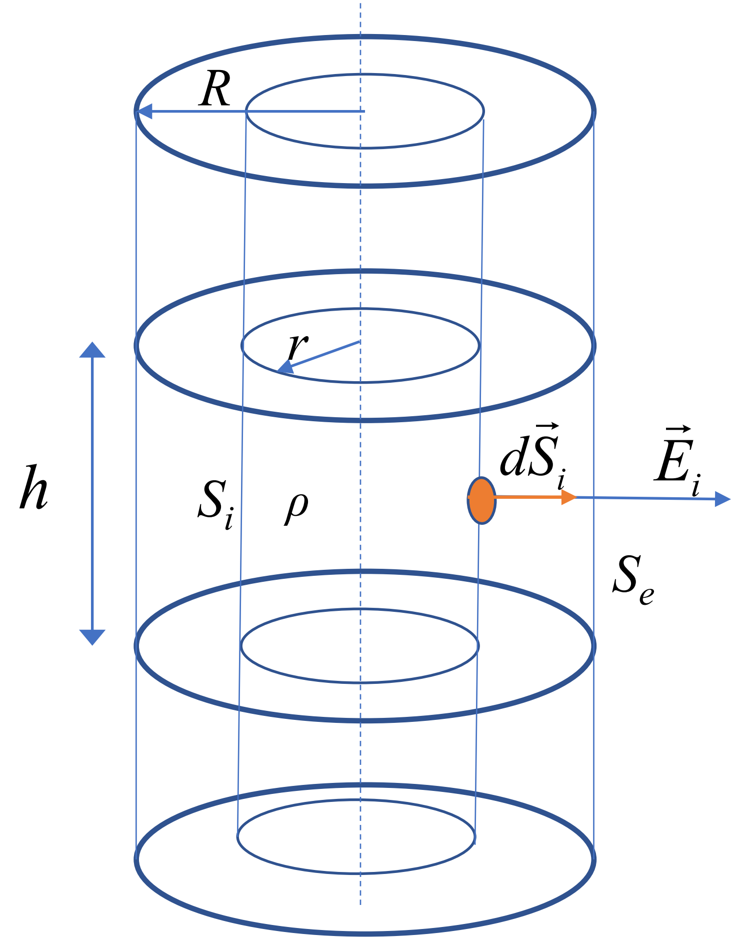
$$E_i \cdot 4\pi r^2 = \frac{\rho \cdot \frac{4\pi r^3}{3}}{\varepsilon_0} \Rightarrow E_i = \frac{\rho \cdot r}{3\varepsilon_0} = \frac{3Q}{4\pi R^3}\frac{r}{3\varepsilon_0} = \frac{Q}{4\pi R^3}\frac{r}{\varepsilon_0}$$

b) $r > R$ ⇨ the field outside the sphere, E_e

$$\oiint_{S_e} \vec{E}_e d\vec{S}_e = \frac{Q}{\varepsilon_0} \Rightarrow E_e \oiint_{S_e} dS_e = \frac{Q}{\varepsilon_0} \Rightarrow E_e \cdot 4\pi r^2 = \frac{Q}{\varepsilon_0} \Rightarrow E_e = \frac{Q}{4\pi\varepsilon_0 \cdot r^2}$$

Can you plot the field against r?

3. A very long dielectric cylinder, of radius *R* and electric permittivity *ε* is charged with a uniform charge density *ρ*. Calculate the electric field as a function of distance *r* from the central axis of the cylinder. **Hint**. Use the Gauss electrostatic theorem



a) $r \le R$ ⇨ the field inside the cylinder, E_i

$$\oiint_{S_i} \vec{E}_i d\vec{S}_i = \frac{Q_i}{\varepsilon_0} \Rightarrow E_i \oiint_{S_i} dS_i = \frac{Q_i}{\varepsilon_0};$$

$$E_i \cdot 2\pi r \cdot h = \frac{\rho \cdot \pi r^2 h}{\varepsilon_0} \quad \Rightarrow \quad E_i(r) = \frac{\rho \cdot r}{2\varepsilon_0};\ E_i(R) = \frac{\rho R}{2\varepsilon_0}$$

b) $r > R$ ⇨ the field outside the cylinder, E_e

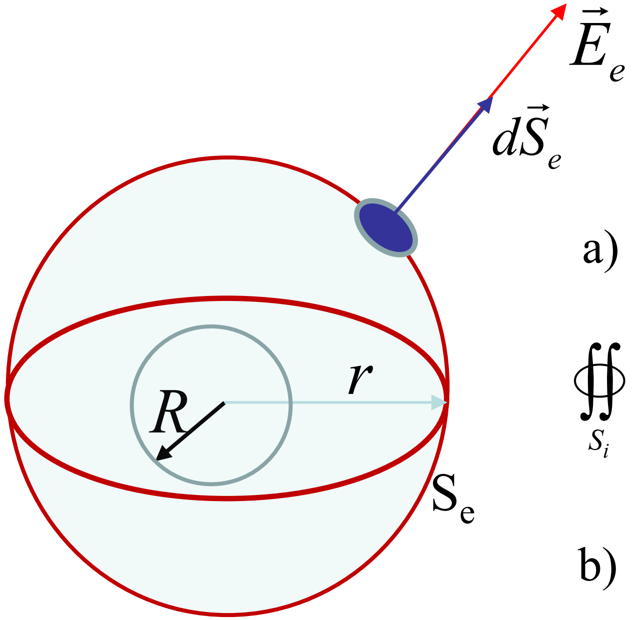
$$\oiint_{S_e} \vec{E}_e d\vec{S}_e = \frac{Q}{\varepsilon_0} \Rightarrow E_e \oiint_{S_e} dS_e = \frac{Q}{\varepsilon_0};$$

$$E_e \cdot 2\pi r \cdot h = \frac{\rho \cdot \pi R^2 h}{\varepsilon_0} \quad \Rightarrow \quad E_e(r) = \frac{\rho \cdot R^2}{2 \cdot r \cdot \varepsilon_0};$$

when r → R, $E_e(R) = \dfrac{\rho \cdot R}{2 \cdot \varepsilon_0}$

Can you plot the field against r?

4. Using the Gauss electrostatic theorem, find the field strength and the potential, in any point, generated by a metallic sphere of radius *R* charged with charge *Q* uniformly distributed



There are no volume charges inside a conductor, the charges are distributed over the surface

a) $r \le R$ ➡ the field inside the sphere, E_i

$$\oiint_{S_i} \vec{E}_i d\vec{S}_i = \frac{Q_i}{\varepsilon_0} \text{ as } Q_i = 0 \;\Rightarrow\; E_i \oiint_{S_i} dS_i = \frac{Q_i}{\varepsilon_0} = 0 \;\Rightarrow\; E_i = 0$$

b) $r > R$ ➡ the field outside the sphere, E_e

$$\oiint_{S_e} \vec{E}_e d\vec{S}_e = \frac{Q}{\varepsilon_0} \Rightarrow E_e \oiint_{S_e} dS_e = \frac{Q}{\varepsilon_0} \Rightarrow E_e \cdot 4\pi r^2 = \frac{Q}{\varepsilon_0} \Rightarrow E_e = \frac{Q}{4\pi\varepsilon_0 \cdot r^2}$$

The potential V(r) can be obtained by integrating the equation

$$\vec{E} = -grad\, V = -\nabla V = -\left(\vec{i}\frac{\partial V}{\partial x} + \vec{j}\frac{\partial V}{\partial y} + \vec{k}\frac{\partial V}{\partial z}\right) \Rightarrow E(r) = -\frac{dV}{dr}$$

for spherical symmetry

Cont.

- when $r \geq R$

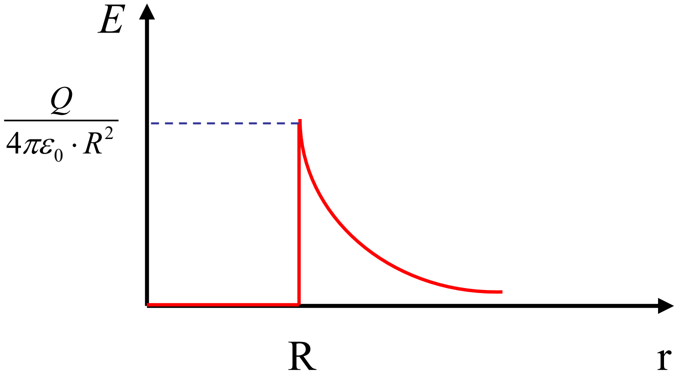
$$E(r) = -\frac{dV}{dr}\ ; \quad V(r) = -\frac{q}{4\pi\varepsilon_0}\int_r^{\infty}\frac{dr}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}\ ; \text{ when r} \rightarrow \text{R} \ \Rightarrow \ V(R) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R}$$

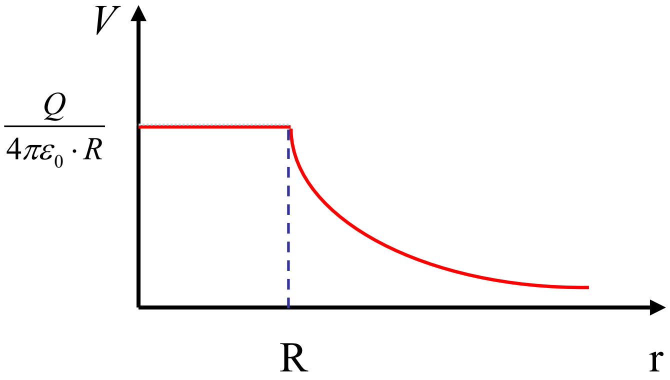
- when r<R

$$E(r) = 0$$

from $E(r) = -\dfrac{dV}{dr} = 0 \Rightarrow V(r) = ct.$

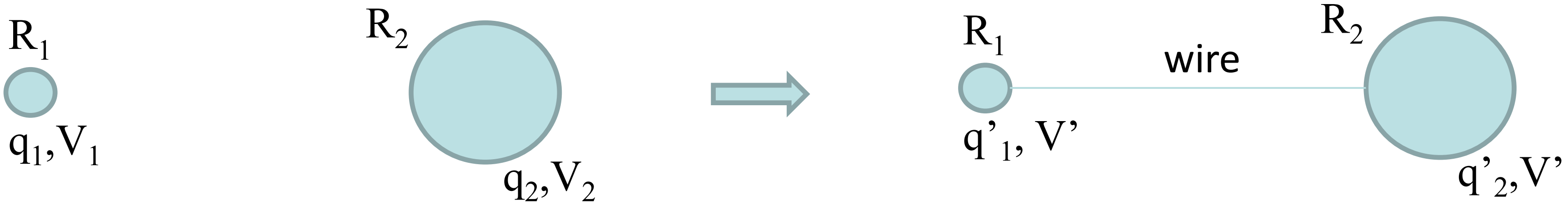
but when r → R, $V(R) = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{r}$ ➡ $V(r) = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{R} = \text{ct.}$





5. Two metallic spheres with R_1=1 cm and R_2=20 cm, having the potentials V_1=9000 V and V_2=900 V, are placed in air at a large distance between them such that don't feel reciprocal interactions. Calculate the potential V' of the system after connecting the spheres through a thin metal wire.

The potential of a sphere of radius R, charged with charge q is: $V(R)=\dfrac{q}{4\pi\varepsilon_0 R}$



before connecting

after connecting the spheres, the potentials will be the same

$$V'=\frac{q'_1}{4\pi\varepsilon_0 R_1}=\frac{q'_2}{4\pi\varepsilon_0 R_2}$$

$$V_1=\frac{q_1}{4\pi\varepsilon_0 R_1}$$

$$V_2=\frac{q_2}{4\pi\varepsilon_0 R_2}$$

$$\Rightarrow$$

$$q_1+q_2=V_1\left(4\pi\varepsilon_0 R_1\right)+V_2\left(4\pi\varepsilon_0 R_2\right)$$
$$q_1+q_2=4\pi\varepsilon_0\left(V_1R_1+V_2R_2\right)$$

$$q'_1+q'_2=V'\left(4\pi\varepsilon_0 R_1\right)+V'\left(4\pi\varepsilon_0 R_2\right)$$
$$q'_1+q'_2=4\pi\varepsilon_0\left(R_1+R_2\right)V'$$

From charge conservation law: $q_1+q_2=q'_1+q'_2$

$$4\pi\varepsilon_0\left(V_1R_1+V_2R_2\right)=4\pi\varepsilon_0\left(R_1+R_2\right)V' \Rightarrow V'=\frac{\left(V_1R_1+V_2R_2\right)}{R_1+R_2}=\frac{9000\cdot 1+900\cdot 20}{1+20}\simeq 1285,71\text{ V}$$

6. Using the Gauss electrostatic theorem, find the field generated by an infinite plane with surface charge density σ.

7. Calculate the electrical potential generated, by a disk with radius R, which carries a charge Q uniform distributed on the surface, in a point situated on the symmetry axis at a distance d above the surface.

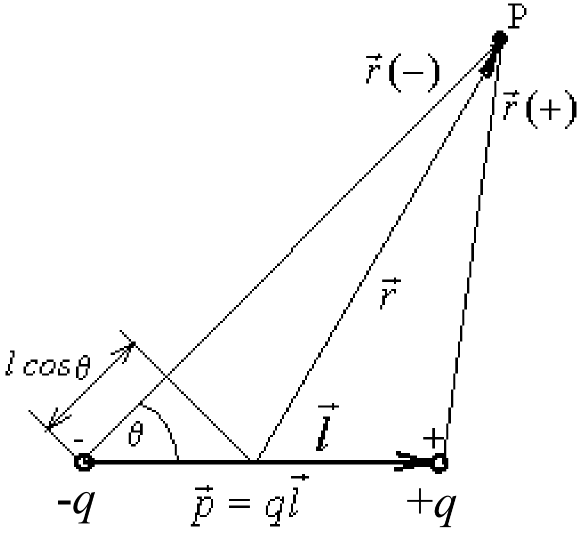
8. Using the Gauss electrostatic theorem, find the field strength in any point generated by a metallic sphere, of radius R_0 charged with charge Q uniformly distributed. The sphere is surrounded by a hollow spherical shell with R_i and R_e (internal and external radius, respectively), a) connected to ground and b) not connected to ground; give a brief discussion on the results and consequences.

9. A dielectric sphere of radius R, and dielectric permittivity ε_0, contains electric charge with volume charge density of the form $\rho = \rho_0 \frac{r}{R}$. Calculate the electric field as a function of distance r from the centre of the sphere, both inside and outside the sphere. **Consider the draw from Problem 2.**

1.8. Electrostatic field in the presence of a dielectric

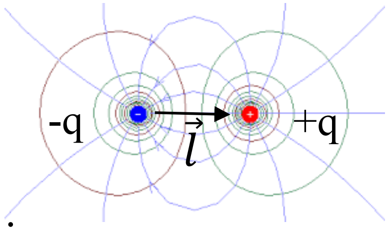
1.8.1. The dipole moment

Consider a pair of positive and negative charges, +q and –q as in the figure bellow. One can define the "**dipole moment**" of this system by:



$$\boxed{\vec{p} = q\vec{l}}$$

dipole moment



The potential generated in point p is:

$$V(P) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_{(+)}} - \frac{1}{r_{(-)}}\right) = \frac{q}{4\pi\varepsilon_0}\left(\frac{r_{(-)} - r_{(+)}}{r_{(+)}r_{(-)}}\right)$$

- because $l \ll r$ ➔ $r_{(-)} - r_{(+)} \approx l\cos\theta$ and $r_{(-)}r_{(+)} \approx r^2$ $\quad l \sim$ nm

$$\boxed{V(p) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3}}$$

where $ql \cdot \cos\theta = (\vec{p}\vec{r})/r$

The electric field strength generated by the dipole:

$$\boxed{\vec{E} = -grad\,V = \frac{1}{4\pi\varepsilon_0}\left[\frac{3(\vec{p}\vec{r})\vec{r}}{r^5} - \frac{\vec{p}}{r^3}\right]}$$

For a dipole: $E \sim \frac{1}{r^3}$ and $V \sim \frac{1}{r^2}$

The strength of the electric field generated by a dipole decreases in inverse proportion to the third power of the distance, i.e. more rapidly than the Coulomb field of a point charge

1.8.2. Polarization of dielectrics

The atoms and molecules can be assumed as tiny dipoles. An external electric field tends to displace positive charges in the direction of the field and the negative charges in the opposite direction. Consequently, the dielectric acquires a ***net dipole moment***. This process is called **polarization** which can be defined by:

$$\boxed{\vec{P} = \frac{\Delta\vec{p}}{\Delta V}}$$ dielectric polarization

$\Delta\vec{p}$ *is the net dipole moment* inside the volume ΔV

1.7.3. Molecular pattern of polarization

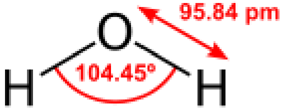
a. Nonpolar atoms and molecules

- atomic or diatomic molecules consisting of identical atoms: He, H_2, O_2, N_2, …
- symmetric polyatomic molecules: CO_2, CH_4, …

In the absence of an external electric field, such a dielectric is **not polarized**

b. Polar atoms and molecules - they possess an electric dipole moment in the absence of an external electric field.

- CO, N_2O, H_2O, SO_2, etc.



c. Ionic lattice polarization - ionic crystals like NaCl, KCl, etc.

1.8.3. Dependence of polarization on the electric field strength

- For most of dielectrics P=0 when E=0
- In *electrets* and *ferroelectrics*, in most cases, P≠0 when E=0.

The electrets and ferroelectrics can maintain their polarization when E=0.
- Barium titanate, Quartz, etc.

- in the general case, the dependence of polarization on the field strength can be expressed as:

$$P_i = \varepsilon_0 \sum_j \chi_{ij} E_j + \varepsilon_0 \sum_{j,k} \chi_{ijk} E_j E_k + \ldots \quad i, j, k = x, y, z$$

- the dielectric is called ***nonlinear***

or

$$P_i = \varepsilon_0 \sum_j \chi_{ij} E_j$$

- the dielectric is called **linear**

If the properties of such a dielectric are different over different directions, the dielectric is called **anisotropic**

The set of 9 quantities χ_{ij} constitutes the **dielectric susceptibility tensor**

For a linear isotropic dielectric

$$\boxed{\vec{P} = \chi \varepsilon_0 \vec{E}}$$

χ is the dielectric susceptibility

Substance	χ
Helium, He	65×10^{-6}
Hydrogen, H_2	254×10^{-6}
Carbon dioxide, CO_2	922×10^{-6}
Water	80
Alcohol	25-30
Transformer oil	2.24
Glass (ordinary)	5
Sodium chloride	5.62
Titanium dioxide	170
Quartz, Barium titanate (ferroelectric)	10^3-10^4

The role of polarization - a separation of positive and negative charges, leading to the appearance of charges in the volume and on the surface of the dielectric. These charges are called **polarization charges** or **bound charges** - they attached to different places in the dielectric and **cannot move freely** in its volume or on its surface.

Bound charges give rise to an electric field in the same way as free charges, and are in no way different from them in this respect.

So, the **presence of a dielectric** is taken into account **by considering** the electric field created by the bound charges induced as a result of polarization.

The electret

Electret (formed of elektr- from "electricity" and -et from "magnet") is a dielectric material that has a *quasi-permanent* ***electric charge*** or ***dipole polarization***.

An electret generates internal and external electric fields, and is the electrostatic equivalent of a permanent magnet. This can be exploited in various applications.

There are two types of electrets:

- **Real-charge electrets** which contain excess charge of one or both polarities, either

- on the dielectric's surfaces (a surface charge)

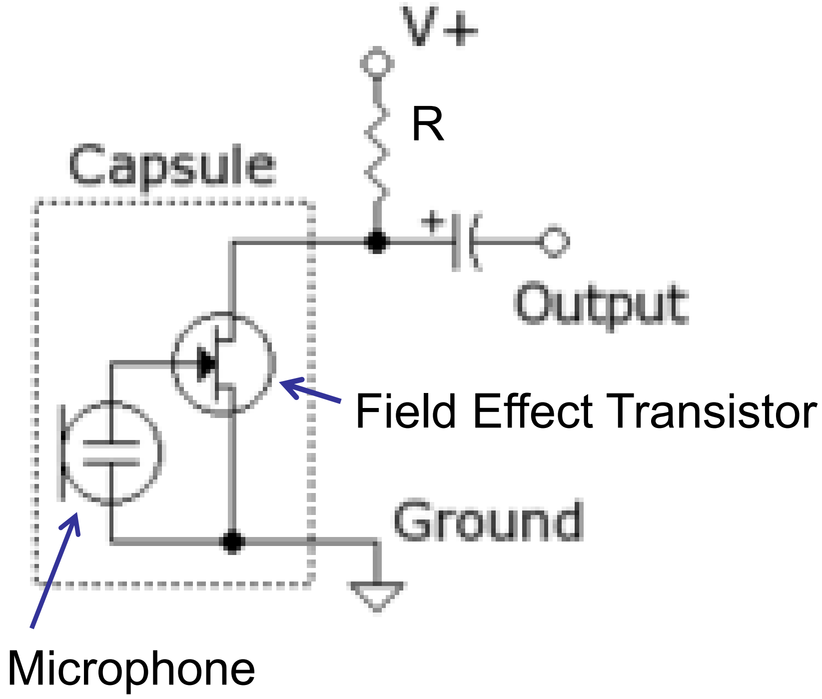
- within the dielectric's volume (a space charge)

- **Oriented-dipole electrets** contain oriented (aligned) dipoles. Ferroelectric materials are one variant of these.

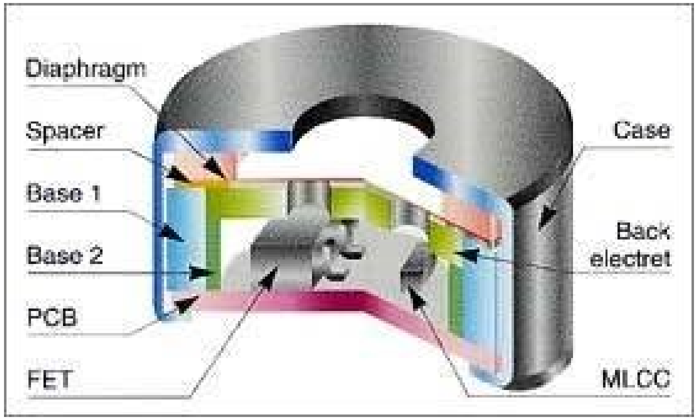
Electret materials: *Quartz* and other forms of *silicon dioxide*, are naturally occurring electrets. Today, most electrets are made from synthetic polymers, e.g. fluoropolymers, polypropylene, polyethyleneterephthalate, etc.

Bulk electrets can be prepared by cooling a suitable dielectric material within a strong electric field (kilovolts/cm), after heating it above its melting temperature. The field repositions the charge carriers or aligns the dipoles within the material. When the material cools, solidification freezes them in position.

Application: Electret microphone



Electret microphones – the capsule



Internal structure of an electret microphone

1.8.4. The electric displacement vector

When a dielectric is present, the equation $div\vec{E} = \rho/\varepsilon_0$ becomes:

$$\boxed{div\vec{E} = \frac{\rho}{\varepsilon_0} + \frac{\rho_b}{\varepsilon_0}}$$

where

ρ - free charges volume density

ρ_b - bound charges volume density

It is shown that: $\rho_b = -div\vec{P}$

$$div\left(\varepsilon_0\vec{E} + \vec{P}\right) = \rho \quad \Rightarrow \quad \boxed{\vec{D} = \varepsilon_0\vec{E} + \vec{P}}$$

- **electric displacement vector**

$$div\vec{D} = \rho$$

- it takes into account the polarization of the medium

Because $\vec{P} = \chi\varepsilon_0\vec{E}$

$$\boxed{\vec{D} = \left(\varepsilon_0 + \chi\varepsilon_0\right)\vec{E} = \varepsilon\,\vec{E}} \qquad \boxed{\varepsilon = \varepsilon_0(1+\chi)}$$

- dielectric constant or permittivity

$$\varepsilon_r = 1 + \chi = \varepsilon/\varepsilon_0$$

- relative permittivity

Gauss electrostatic theorem in the presence of dielectrics becomes:

$$\int_V div\vec{D}\cdot dV = \int_V \rho\cdot dV \quad \Rightarrow \quad \boxed{\oint_S \vec{D}d\vec{S} = Q}$$

Q represents the total charge inside the volume, *V*

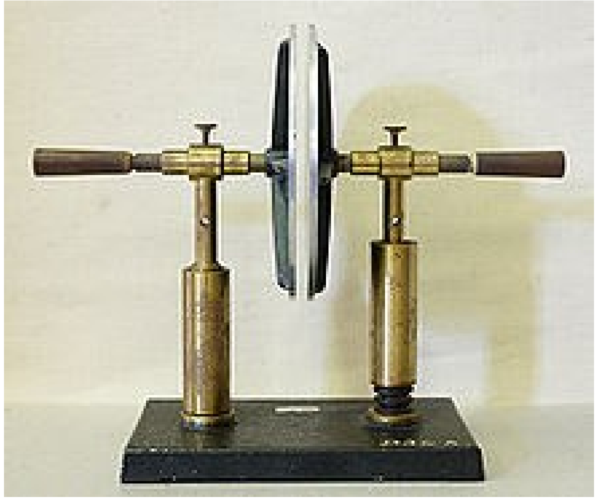
1.9. The capacitor

The capacitor is a passive electronic component consisting of a pair of conductors (plates) separated by a dielectric (insulator).

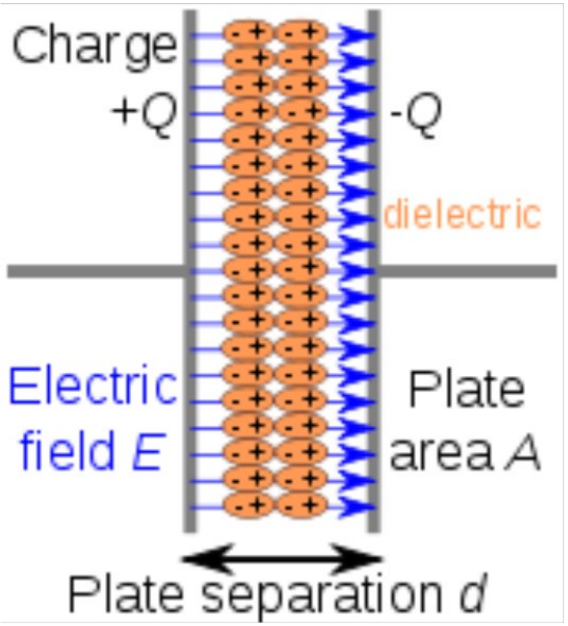
When there is a potential difference (voltage) between the plates, a static electric field develops in the dielectric that stores energy and produces a mechanical force between the plates. An **ideal capacitor** is characterized by a single constant value, **capacitance**, measured in **Farads**. This is the ratio of the electric charge on each conductor to the potential difference between them.

In October 1745, Ewald Georg von Kleist of Pomerania in Germany found that charge could be stored by connecting a high voltage electrostatic generator by a wire to a volume of water in a hand-held glass jar.
Von Kleist's hand and the water acted as conductors and the jar as a dielectric

Battery of four Leyden jars in Museum Boerhaave, Leiden, the Netherlands



A parallel-plate capacitor



Charge separation in a parallel-plate capacitor causes an internal electric field. A dielectric (orange) reduces the field and increases the capacitance.

The conductors hold equal and opposite charges, Q, on their facing surfaces and develops an electric field and hence a potential difference, U.

We can define the **capacitance** by: $C = \frac{Q}{U} \quad \left[\frac{1C}{1V} = 1F\right]$, in SI

Sometimes charge build-up affects the capacitor mechanically, causing its capacitance to vary. In this case, capacitance is defined in terms of incremental changes $C = dQ/dU$

1.9.1. Energy of the electrostatic field

Energy of interaction between discrete charges

$$W = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{r} \qquad W = \frac{1}{2}\left(\frac{1}{4\pi\varepsilon_0}\frac{Q_2}{r}Q_1 + \frac{1}{4\pi\varepsilon_0}\frac{Q_1}{r}Q_2\right) = \frac{1}{2}\left(V_1 Q_1 + V_2 Q_2\right)$$

with $V_1 = \frac{1}{4\pi\varepsilon_0}\frac{Q_2}{r}$, $V_2 = \frac{1}{4\pi\varepsilon_0}\frac{Q_1}{r}$

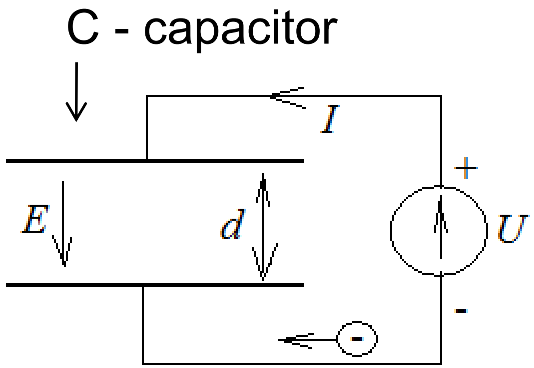
This formula can be easily generalised for the case of small several charged spheres

$W = \frac{1}{2}\sum_i V_i Q_i$ or, for a continuous distribution of charges: $W = \frac{1}{2}\iiint V\rho d\mathrm{v}$

Energy density of an electric field

The work performed by the source to bring the charge dq on the capacitor plates is:

$$dW = U\cdot dq \Rightarrow dW = C\cdot U\cdot dU \Rightarrow \boxed{W = C\int_0^U U dU = \frac{1}{2}CU^2}$$



This work to charge the capacitor is found as stored energy, W_E.

$$C = \frac{\varepsilon_0\varepsilon_r S}{d} \Rightarrow W_E = V\cdot\frac{1}{2}\vec{E}\vec{D} = w_E\cdot V;\; V = S\cdot d$$

$\boxed{w_E = \frac{1}{2}\vec{E}\vec{D}}$ Energy density of the electric field

Supercapacitors

There is a special type of capacitors with capacitances larger than 100 F. They are used mainly for energy storage.

Example:

If a capacitor with C=100 F is charged at U=100 V, then, the stored energy, W_E is:

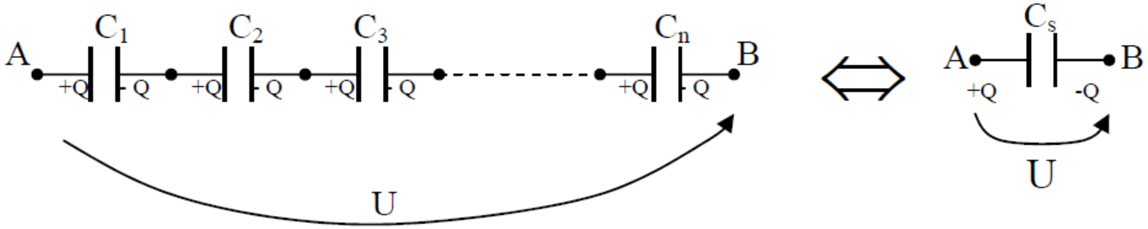
$$W_E = \frac{1}{2} C U^2 = \frac{1}{2} 100 \cdot 10^4 = 500 \text{ kJ}$$

- supercapacitors have a capacitance up to 3000 farad;
- they have a much higher power density than batteries or fuel cells;

Applications: vehicles, complementing batteries, alternative energy (replacing batteries with capacitors in conjunction with novel energy sources)

1.9.2. Connections of capacitors

Series connection of capacitors



$$U = \frac{Q}{C_S} \quad \text{and} \quad U = U_1 + U_2 + \ldots + U_n = \frac{Q}{C_1} + \frac{Q}{C_2} + \ldots + \frac{Q}{C_n} = \frac{Q}{C_S}$$

$$\boxed{\frac{1}{C_S} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_n} \quad \Rightarrow \quad \frac{1}{C_S} = \sum_i \frac{1}{C_i}}$$

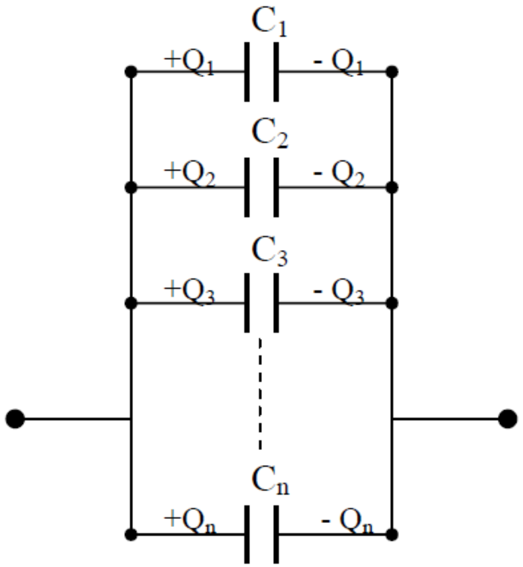
Parallel connection of capacitors

Q – total charge

$$Q = Q_1 + Q_2 + \ldots + Q_n \;\Rightarrow\; C_P U = C_1 U + C_2 U + \ldots + C_n U$$

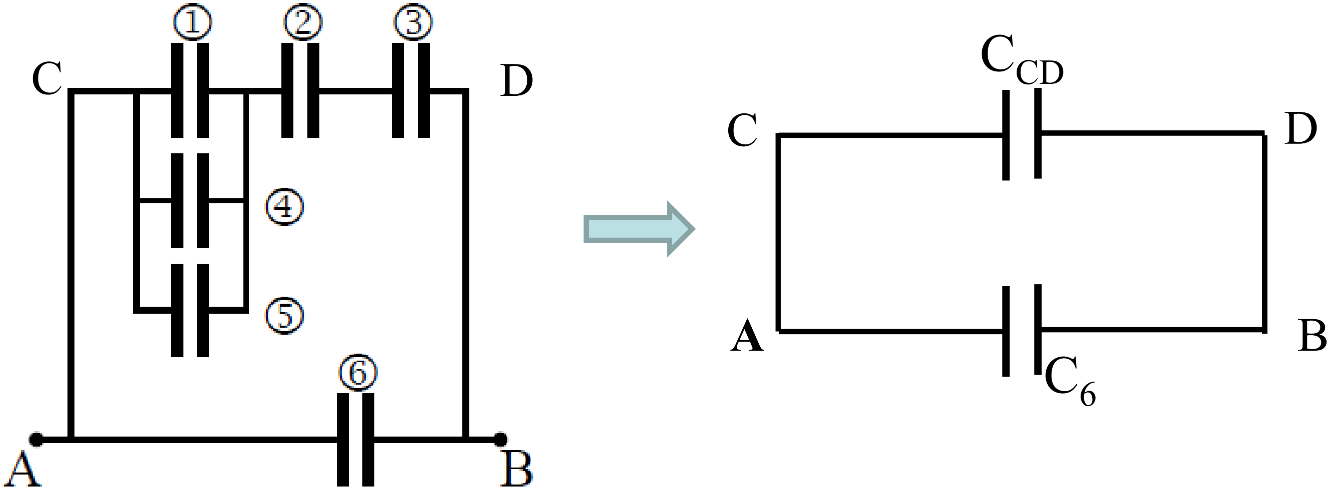
$$C_P = C_1 + C_2 + \ldots + C_n$$

$$\boxed{C_P = \sum_i C_i}$$



1.10. Solved and proposed applications – part 3

1. Calculate the equivalent capacitance of the system, where C=70 nF:

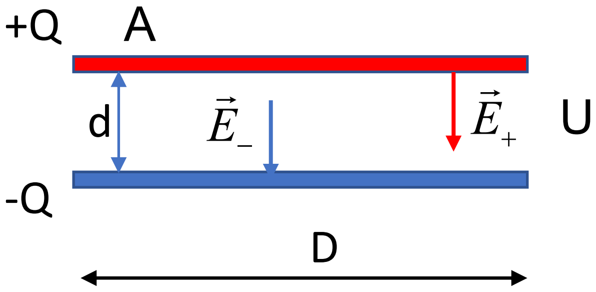


$$C_{145} = C_1 + C_4 + C_5 = C + C + C = 3C$$

$$\frac{1}{C_{23}} = \frac{1}{C_2} + \frac{1}{C_3} = \frac{1}{C} + \frac{1}{C} = \frac{2}{C} \Rightarrow C_{23} = \frac{C}{2}$$

$$\frac{1}{C_{CD}} = \frac{1}{C_{145}} + \frac{1}{C_{23}} = \frac{1}{3C} + \frac{2}{C} = \frac{7}{3C} \Rightarrow C_{CD} = \frac{3C}{7} \Rightarrow C_{echiv} = C_{CD} + C_6 = \frac{3C}{7} + C = \frac{10C}{7}$$

2. Obtain the capacitance, *C*, for a planar parallel-plate capacitor.



- the field generated by an infinite plane with surface charge density σ:

$$E_+ = \frac{\sigma}{2\varepsilon_0} \quad E_- = -\frac{\sigma}{2\varepsilon_0} \Rightarrow E = E_+ - E_- \Rightarrow E = \frac{\sigma}{\varepsilon_0}$$

Electric field between the plates is uniform ➔ U=E×d

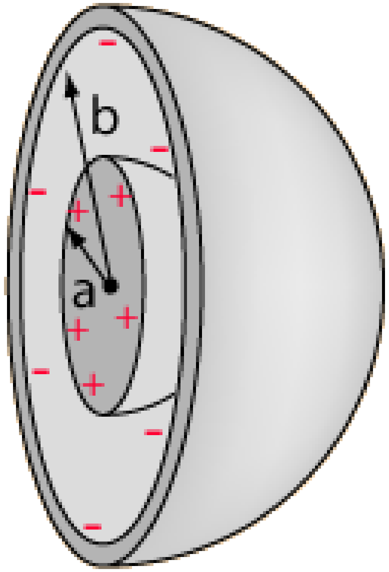
$$U = E \times d = \frac{\sigma}{\varepsilon_0} \cdot d$$

Assuming the plate area: A ➔ $U = \frac{\sigma}{\varepsilon_0} \cdot \frac{d \cdot A}{A} = \frac{Q \cdot d}{\varepsilon_0 A}$

From the definition of capacitance: $C = \frac{Q}{U} \Rightarrow C = \frac{Q}{U} = \frac{\varepsilon_0 A}{d}$

If between the plates is a dielectric with ε_r: $C = \frac{\varepsilon_0 \varepsilon_r A}{d}$

3. Calculate the capacitance of a spherical capacitor whose plates have radii *a* and *b*, with vacuum between them; calculate the capacitance of an isolated charged conducting sphere of radius *R*.



By applying Gauss' law to a charged conducting sphere, the **electric field outside** it is found to be:

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}\ ,\ r > a$$

The voltage between the spheres can be found by integrating the electric field along a radial line:

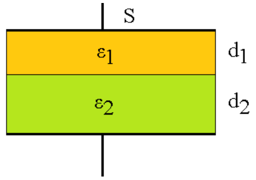
$$\Delta V = U = \frac{Q}{4\pi\varepsilon_0}\int_a^b \frac{1}{r^2}\,dr = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)$$

From the definition of capacitance: $C = \frac{Q}{\Delta V} = \frac{4\pi\varepsilon_0}{\left[\frac{1}{a}-\frac{1}{b}\right]}$

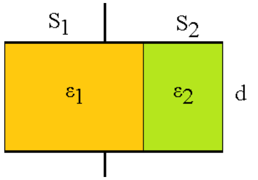
The capacitance of an isolated charged conducting sphere of radius R can be found from:

$$b \to \infty, \text{and } a = R \Rightarrow C_{sph} = 4\pi\varepsilon_0 R$$

4. A parallel-plate capacitor contains two dielectrics with electrical permitivities ε_1 and ε_2 and thicknesses d_1 and d_2 as in the figure. Knowing the plate area S, find the capacitance C of the capacitor.



5. A parallel-plate capacitor contains two dielectrics with electrical permitivities ε_1 and ε_2 and surface areas S_1 and S_2 as in the figure. Knowing the interplate distance d, find the capacitance C of the capacitor.



6. Find the boundary conditions for $\vec{E}$ and $\vec{D}$ at the interface (separation surface) between two dielectric media with electrical permitivities ε_1 and ε_2 .

a) The tangential component of $\vec{E}$ is continuous at the <u>separation surface</u>

$$\oint_L \vec{E}d\vec{l} = 0 \Rightarrow \int_{L_1} \vec{E}_1 d\vec{l}_1 + \int_{L_2} \vec{E}_2 d\vec{l}_2 = \int_{L_1} \left(E_{1t} - E_{2t}\right) dl_1 = 0$$

because $d\vec{l}_1 = -d\vec{l}_2$

⇨ $\boxed{E_{1t} = E_{2t} \Rightarrow \frac{D_{1t}}{\varepsilon_{r1}} = \frac{D_{2t}}{\varepsilon_{r2}}}$

b) The normal component of $\vec{D}$ is continuous at the <u>separation surface</u>

$$\oint\!\!\!\oint_S \vec{D}d\vec{S} = Q \Rightarrow \iint_{S_1} \vec{D}_1 d\vec{S}_1 + \iint_{S_2} \vec{D}_2 d\vec{S}_2 = \iint_{S_1} \sigma dS_1 \Rightarrow D_{1n} - D_{2n} = \sigma$$

σ - surface charge density

If σ=0 at the separation surface, then: $\boxed{D_{1n} - D_{2n} = 0 \Rightarrow D_{1n} = D_{2n} \Rightarrow \varepsilon_{r1} E_{1n} = \varepsilon_{r2} E_{2n}}$

7. Find the refraction low for the electric field lines at the interface (separation surface) between two dielectric media with electrical permitivities ε_1 and ε_2 .

1.11. The charge conservation law

a) The integral form of the charge conservation law

By definition: $I = \frac{dQ}{dt}$ [C/s]=[A] $I = \int_S \vec{j}\,d\vec{S}$ j - current density [A/m^2]

For a closed surface: $\frac{\partial Q}{\partial t} = -\oint_S \vec{j}\,d\vec{S}$ ⇨ $\boxed{\frac{\partial}{\partial t}\int_V \rho\,d\mathrm{v} = -\oint_S \vec{j}\,d\vec{S}}$ ρ – volume charge density

b) Differential form of the charge conservation law

$$\oint_S \vec{j}\,d\vec{S} = \int_V div\vec{j}\cdot d\mathrm{v}$$

- from Gauss's theorem $\oiint_S \vec{f}\,d\vec{S} = \iiint_V div\vec{f}\,dV$

⇩

$$\int_V \left(\frac{\partial \rho}{\partial t} + div\vec{j}\right) d\mathrm{v} = 0$$

⇩

$$\boxed{\frac{\partial \rho}{\partial t} + div\vec{j} = 0}$$

- *the differential form of the charge conservation law*
- the **continuity equation**

1.11.1. Electrostatic field in the presence of conductors

In electrostatics, we consider the case when **charges are fixed**, i.e.:

$$\vec{j} = 0 \;\Rightarrow\; \vec{E} = 0$$

⇩

Absence of volume charge inside a conductor

$$\rho = 0$$ ➔ ***there are no volume charges inside a conductor !!***

Both positive and negative charges exist inside the conductor, but they compensate each other, and the interior of the conductor is neutral on the whole.

Suppose that for t=0, we bring charges on conductor such that $\rho(0) \neq 0$

$$\frac{\partial \rho}{\partial t} + div\vec{j} = 0$$

$$\vec{j} = \sigma \cdot \vec{E}$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \sigma \cdot div\vec{E} = 0 \qquad \text{but:} \quad div\vec{E} = \frac{\rho}{\varepsilon_0} \Rightarrow \frac{\partial \rho}{\partial t} = -\sigma \frac{\rho}{\varepsilon_0} \quad (1)$$

σ - el. conductivity

ρ – vol. charge density

By integrating the above equation (1), we find:

$$\boxed{\rho(t) = \rho(0) e^{-\left(\frac{\sigma}{\varepsilon_0}\right) \cdot t}} \;\Rightarrow\; \boxed{\rho(t) \to 0}$$

Calculation steps

$$\frac{\partial\rho}{\partial t}=-\sigma\frac{\rho}{\varepsilon_0}\Rightarrow\frac{d\rho}{\rho}=-\frac{\sigma}{\varepsilon_0}dt\Rightarrow\int_{\rho(0)}^{\rho}\frac{d\rho}{\rho}=-\frac{\sigma}{\varepsilon_0}\int_0^t dt$$

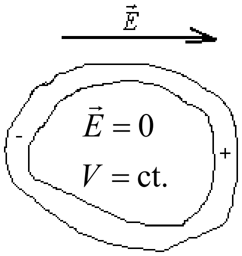
$$\ln\rho-\ln\rho(0)=-\frac{\sigma}{\varepsilon_0}t\Rightarrow\ln\frac{\rho(t)}{\rho(0)}=-\frac{\sigma}{\varepsilon_0}t\Rightarrow\rho(t)=\rho(0)e^{-\frac{\sigma}{\varepsilon_0}t}\Rightarrow\rho(t)\to 0$$

The *space charge* in the conductor is "assimilated" during the time $\boxed{\tau=\frac{\varepsilon_0}{\sigma}}$

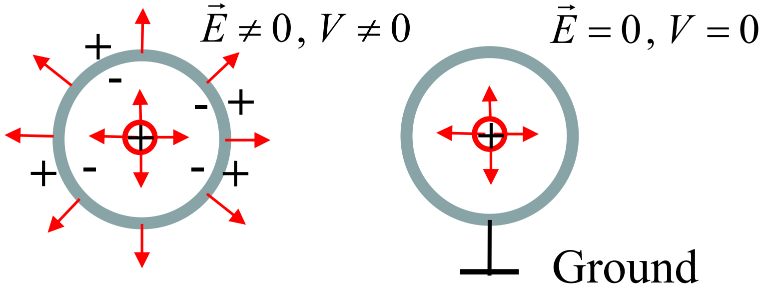
τ is named **relaxation time**; $\tau \sim 10^{-19}$ s for Cu

For moderate frequencies, *free charges in a conductor are distributed over its surface and volume charges are absent*

Metallic screen



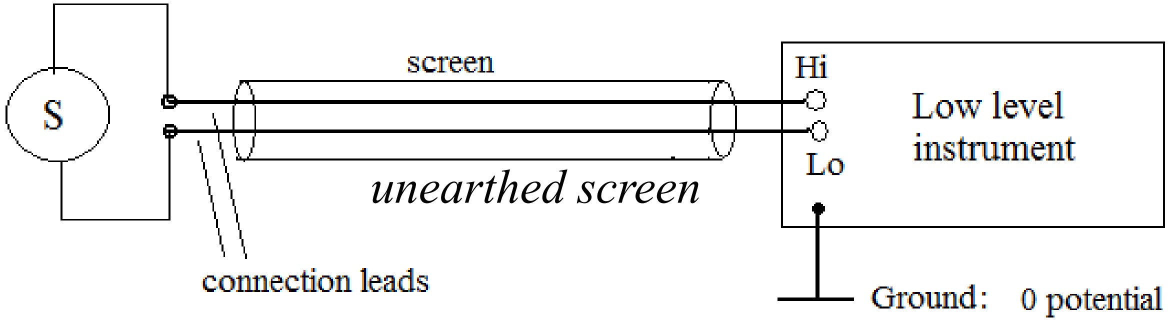
A metallic screen for external fields

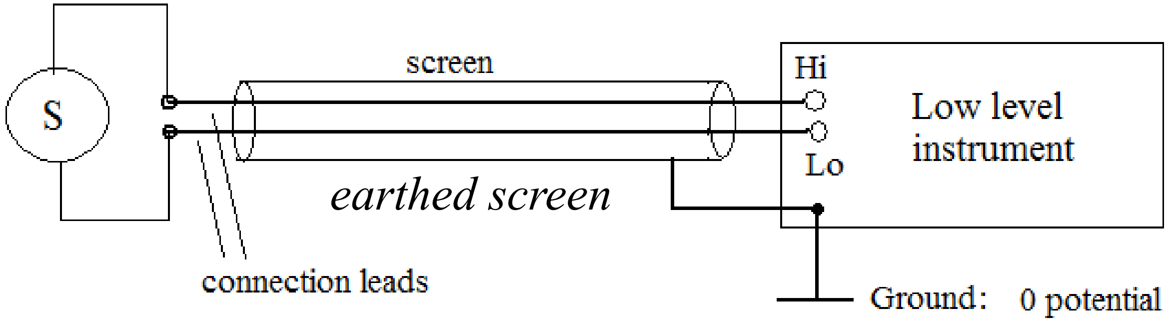


A charge surrounded by a closed conducting shell

The ***earthed closed shell*** *shields the external space from the charges located in the volume surrounded by this shell. An unearthed shell does not provide such a screening*

Screening for low level signals manipulation

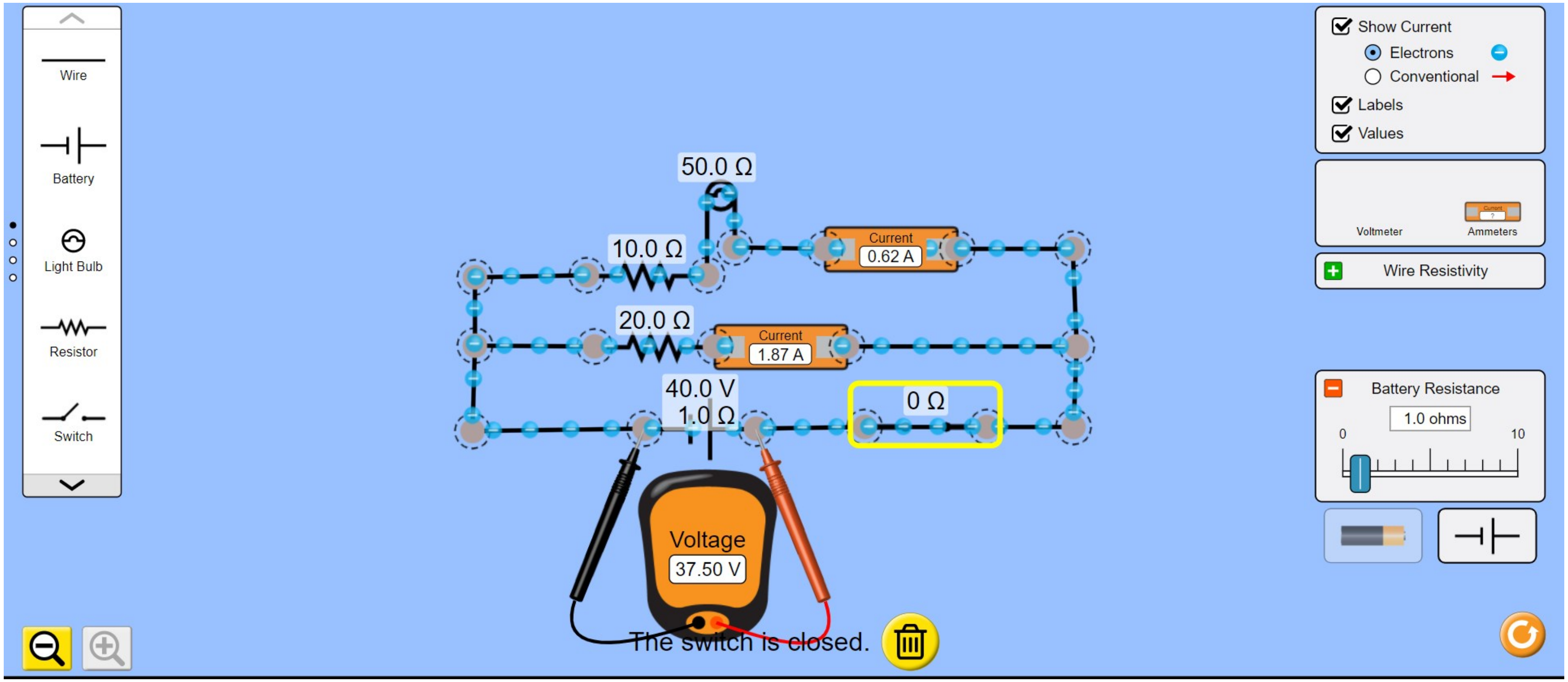




Earthed closed screen *shields the connection leads from the exterior charges and provides immunity to exterior electrical perturbations*

Chapter II. Conductive media in electric fields

Electric current



☑ Show Current
● Electrons
○ Conventional
☑ Labels
☑ Values

Battery Resistance 1.0 ohms

The switch is closed.

Useful web resourses:

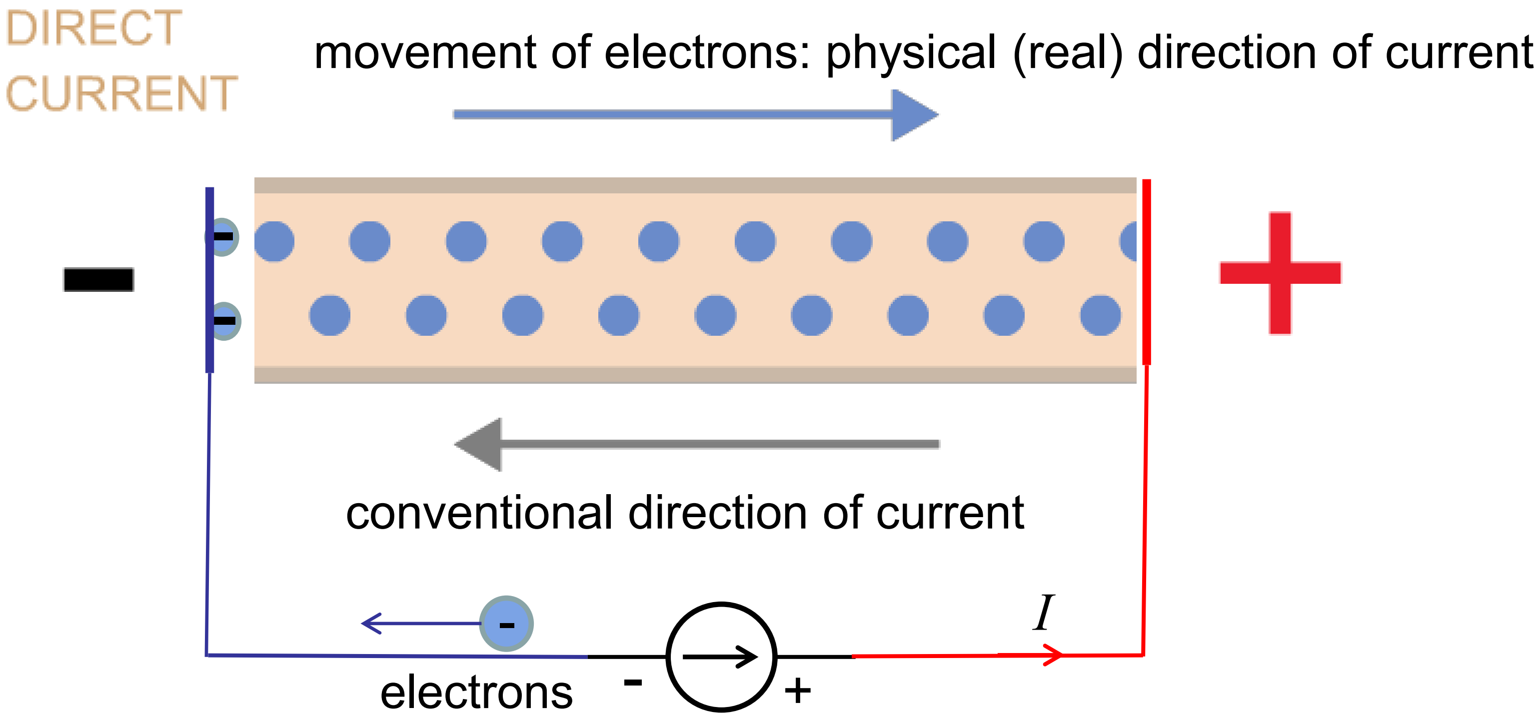
https://phet.colorado.edu/en/simulation/circuit-construction-kit-dc-virtual-lab

https://phet.colorado.edu/en/simulation/circuit-construction-kit-dc

Extended Table of Contents

2.1. Introductory aspects

In electrostatics, E=0 inside conductors.
When charges are moving inside the conductors, ***E is not 0***



Physical and conventional direction of current in a conductor

Current intensity is defined as the charge, Q, which is flowing through the conductor in unit of time ➔ I=Q/Δt [A]=[C/s]

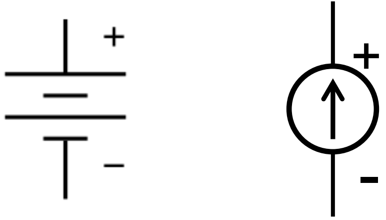
2.1.1. Generation of electric current

Electromotive source (EMF - Electromotive force):

- separates positive from negative charges inside the source
- allows the driving of charges inside the source such that a current can be sustained through an external circuit.

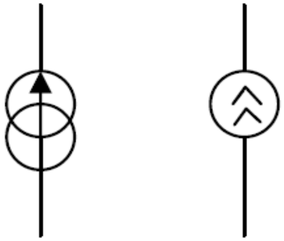
Symbols:

Voltage source – has a constant EMF



Battery EMF Generator

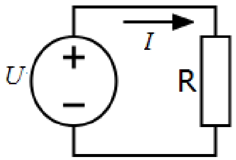
Current source – generates a constant current



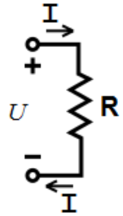
2.1.2. The Ohm's Law

Was discovered experimentally in 1827 by Ohm (1787-1854) → Ohm's Law

Ohm's law states that the current through a conductor between two points is directly proportional to the potential difference across the two points. Introducing the constant of proportionality, the resistance, one arrives at the usual mathematical equation that describes this relationship: $I = U/R$



or

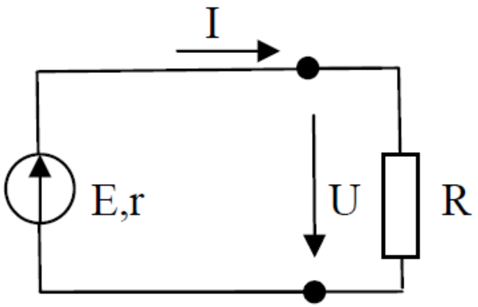


$$\boxed{I = \frac{U}{R}}$$

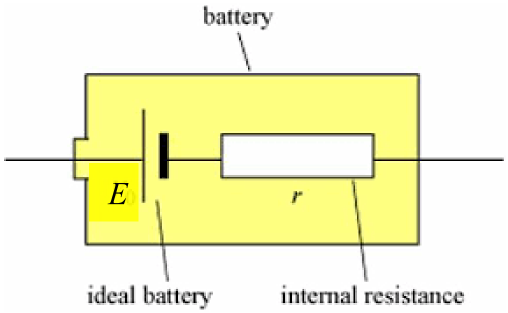
The Ohm's Law

R - resistance [Ω]
1/R – conductance [S]
S – Siemens, Ω - Ohm

Using the emf of the source, *E*, the Ohm's law can be expressed for the entire circuit as:



$$I = \frac{E}{R + r}$$

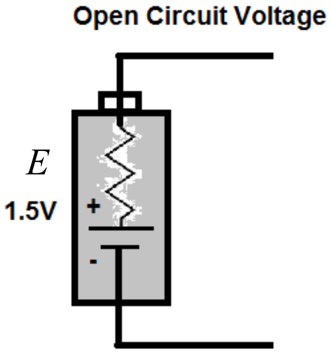


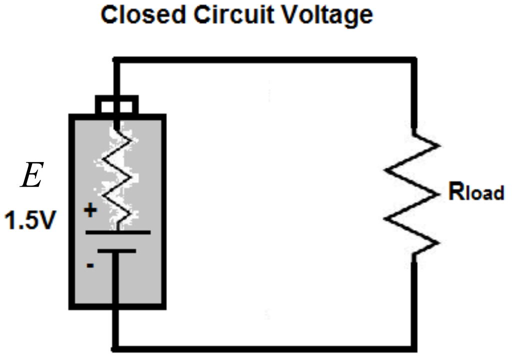
r – internal resistance of a (source) battery

Some basic regimes in electrical circuits

The short circuit current

$$R = 0 \;\Rightarrow\; I_{SC} = \frac{E}{r}$$





Open circuit voltage

When $R \to \infty \;\Rightarrow\; U_{open} = E$

2.2. Kirchhoff's circuit laws

2.2.1. Kirchhoff's current law (KCL)

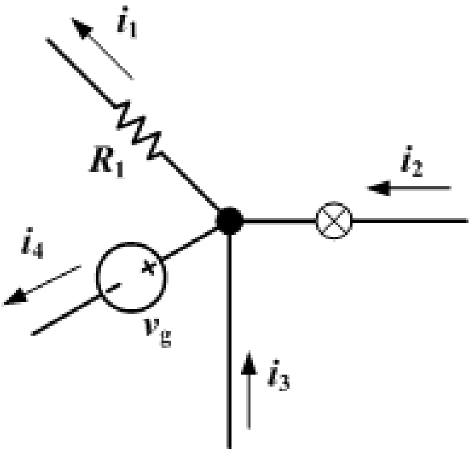
This law is also called Kirchhoff's first law, Kirchhoff's point rule, Kirchhoff's junction rule (or nodal rule), and Kirchhoff's first rule.

The principle of conservation of electric charge implies that:

- at any node (junction) in an electrical circuit, the sum of currents flowing into that node is equal to the sum of currents flowing out of that node, **or:**
- the algebraic sum of currents in a network of conductors meeting at a point is zero.

$$\sum_{k=1}^{n} I_k = 0$$

n is the total number of branches with currents flowing towards or away from the node.



This formula is true, also, for complex currents: $\sum_{k=1}^{n} \hat{I}_k = 0$

The law is based on the conservation of charge whereby the charge (measured in coulombs) is the product of the current (in amperes) and the time (in seconds).

2.2.2. Kirchhoff's voltage law (KVL)

This law is also called **Kirchhoff's second law**, **Kirchhoff's loop (or mesh) rule**, and **Kirchhoff's second rule**.

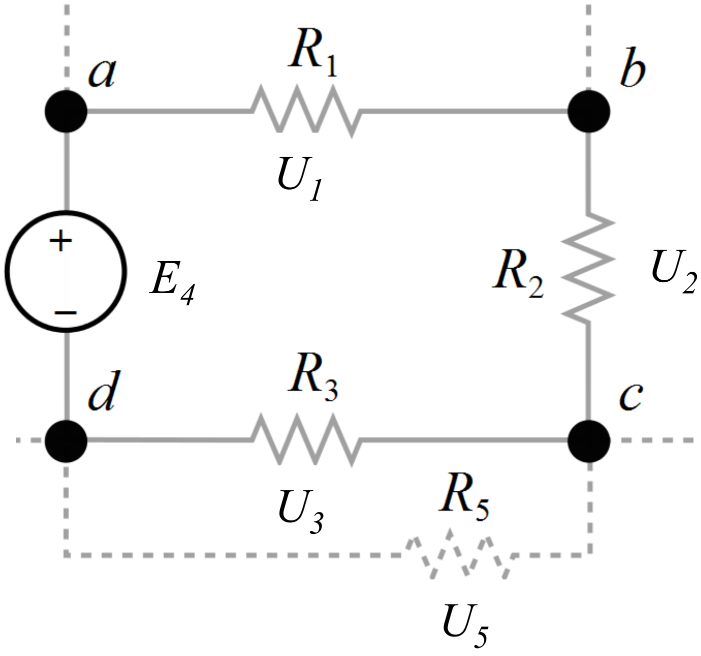
The principle of energy conservation implies that the directed sum of the electrical potential differences (voltage) around any closed circuit is zero, **or**:

More simply, the sum of the emfs in any closed loop is equivalent to the sum of the potential drops in that loop, **or**:

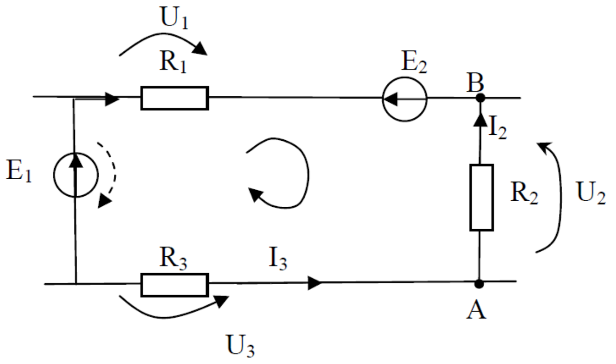
The algebraic sum of the products of the resistances of the conductors and the currents in them in a closed loop is equal to the total emf available in that loop.

Similarly to KCL, it can be stated as:

$$\sum_{k=1}^{n} U_k = 0 \qquad \sum_{k=1}^{n} \hat{U}_k = 0 \text{ for a.c. signals}$$

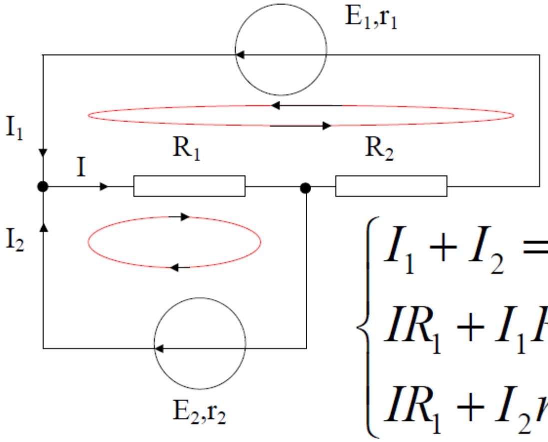


Example 1:



$$\underbrace{E_1 - E_2}_{} = \underbrace{R_1I_1 - R_2I_2 - R_3I_3}_{}$$

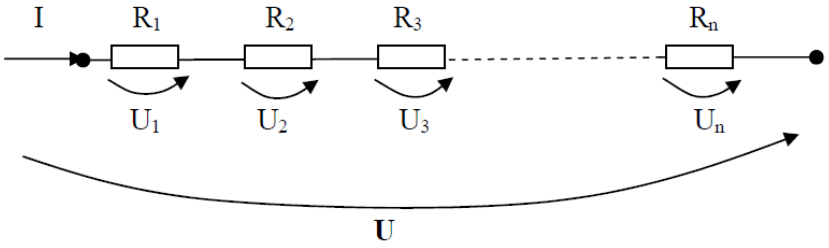
Example 2:



$$\begin{cases} I_1 + I_2 = I \\ IR_1 + I_1R_2 + I_1r_1 = E_1 \\ IR_1 + I_2r_2 = E_2 \end{cases}$$

2.2.3. Applications of Kirchhoff's laws

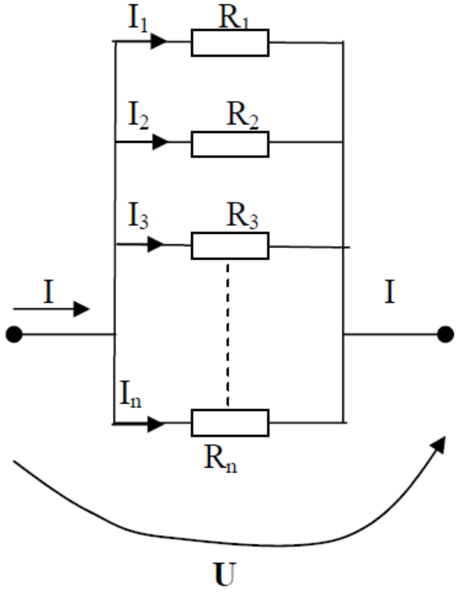
i. Series connection of resistors



$$U = U_1 + U_2 + \ldots + U_n$$

$$I \cdot R_{series} = I \cdot R_1 + I \cdot R_2 + \ldots + I \cdot R_n \;\Rightarrow\; R_{series} = R_1 + R_2 + \ldots + R_n \Rightarrow R_{series} = \sum_{i=1}^{n} R_i$$

ii. Parallel connection of resistors



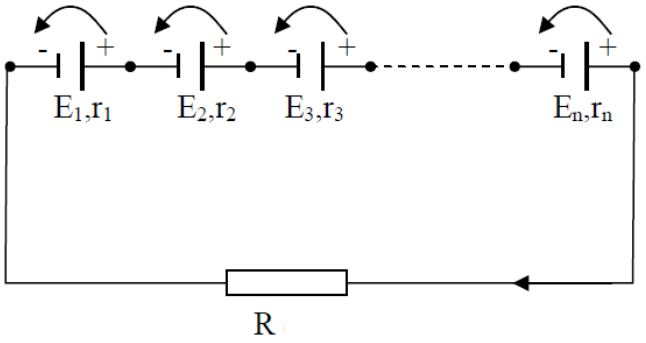
$$I = I_1 + I_2 + \ldots + I_n$$

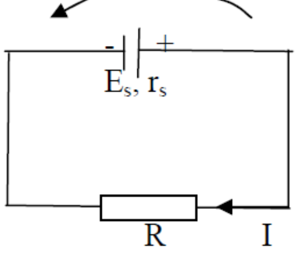
$$\frac{U}{R_{parallel}} = \frac{U}{R_1} + \frac{U}{R_2} + \ldots + \frac{U}{R_n}$$

$$\frac{1}{R_{parallel}} = \frac{1}{R_1} + \frac{1}{R_2} + \ldots + \frac{1}{R_n} \Rightarrow \frac{1}{R_{parallel}} = \sum_{i=1}^{n} \frac{1}{R_i}$$

If: $R_1 = R_2 = \ldots = R_n = R \Rightarrow R_{parallel} = \frac{R}{n}$

iii. Series connection of sources

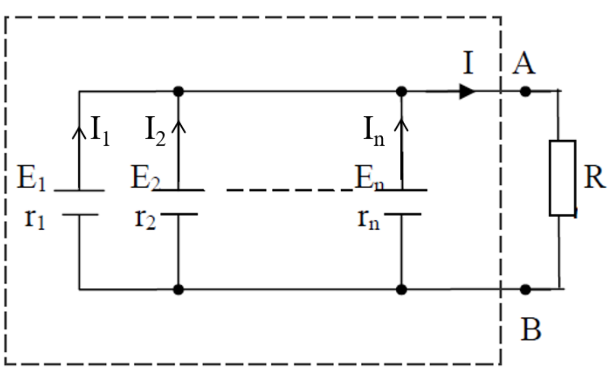


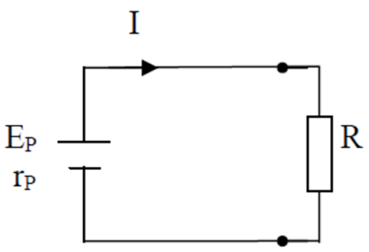


$$E_1 + E_2 + \ldots + E_n = Ir_1 + Ir_2 + \ldots + Ir_n + IR = E_S$$

$r_S = r_1 + r_2 + \ldots + r_n$ and $E_S = E_1 + E_2 + \ldots + E_n$ ⇨ $I = \frac{E_S}{r_S + R}$

iv. Parallel connection of sources





Equivalent circuit

Applying Kirchhoff's laws:

$$I = I_1 + I_2 + \ldots + I_n \quad \text{and} \quad I = \frac{E_P}{r_P + R} \quad (1)$$

$$E_1 = I_1 r_1 + IR \;\Rightarrow\; I_1 = \frac{E_1}{r_1} - I\frac{R}{r_1}$$

$$E_2 = I_2 r_2 + IR \;\Rightarrow\; I_2 = \frac{E_2}{r_2} - I\frac{R}{r_2}$$

....

$$E_n = I_n r_n + IR \;\Rightarrow\; I_n = \frac{E_n}{r_n} - I\frac{R}{r_n}$$

$$\Rightarrow$$

$$I = \frac{E_1}{r_1} + \frac{E_2}{r_2} + \ldots + \frac{E_n}{r_n} - IR\left(\frac{1}{r_1} + \frac{1}{r_2} + \ldots + \frac{1}{r_n}\right)$$

$$I\left(\frac{1}{r_1} + \frac{1}{r_2} + \ldots + \frac{1}{r_n}\right)^{-1} + IR = \left(\frac{E_1}{r_1} + \frac{E_2}{r_2} + \ldots + \frac{E_n}{r_n}\right)\left(\frac{1}{r_1} + \frac{1}{r_2} + \ldots + \frac{1}{r_n}\right)^{-1}$$

$$\Rightarrow I = \frac{\left(\frac{E_1}{r_1} + \frac{E_2}{r_2} + \ldots + \frac{E_n}{r_n}\right)\left(\frac{1}{r_1} + \frac{1}{r_2} + \ldots + \frac{1}{r_n}\right)^{-1}}{\left(\frac{1}{r_1} + \frac{1}{r_2} + \ldots + \frac{1}{r_n}\right)^{-1} + R}$$

$$\boxed{\begin{aligned} E_P &= \left(\frac{E_1}{r_1} + \frac{E_2}{r_2} + \ldots + \frac{E_n}{r_n}\right) \Big/ \left(\frac{1}{r_1} + \frac{1}{r_2} + \ldots + \frac{1}{r_n}\right) \\ r_P &= \left(\frac{1}{r_1} + \frac{1}{r_2} + \ldots + \frac{1}{r_n}\right)^{-1} \qquad (2) \end{aligned}}$$

For n identical sources:

$$I = \frac{n\frac{E}{r}\frac{r}{n}}{\frac{r}{n} + R} = \frac{E}{\frac{r}{n} + R} = \frac{nE}{r + nR}$$

2.3. Differential form of Ohm's law – a simple theoretical model

It can be shown that:

$$\vec{j} = -ne\vec{v}$$

where

- j is the current density [A/m^2]
- n is the volume density of carriers (n=N/V)
- v is the mean velocity (***drift velocity***) [m/s]
- e is the charge of the electron [C]

$$\vec{v}_{\max} = \vec{a}\tau = -\frac{e\vec{E}}{m}\tau$$

where τ is the time between two collisions ~10^{-12} s !
- *relaxation time*

The mean velocity can be simply expressed by:

$$\vec{v} = \frac{\vec{v}_{\max}+0}{2} = -\frac{e\tau}{2m}\cdot\vec{E} = -\mu\cdot\vec{E}$$

where $\boxed{\mu = \frac{e\tau}{2m}}$ is the carrier *mobility*

$$\Downarrow$$

$$\boxed{\vec{j} = \frac{e^2 n\tau}{2m}\cdot\vec{E} = \sigma\cdot\vec{E}}$$

Differential form of Ohm's law

with $\boxed{\sigma = \frac{e^2 n\tau}{2m}}$ [S/m] - the electric conductivity

$\sigma = 1/\rho$ 1 S=1 Ω^{-1}

More general expressions for μ and σ

$$\mu = \frac{e\cdot\langle\tau\rangle}{m^*} \quad \text{and} \quad \sigma = \frac{e^2 n\cdot\langle\tau\rangle}{m^*} \Rightarrow \boxed{\sigma = ne\mu}$$

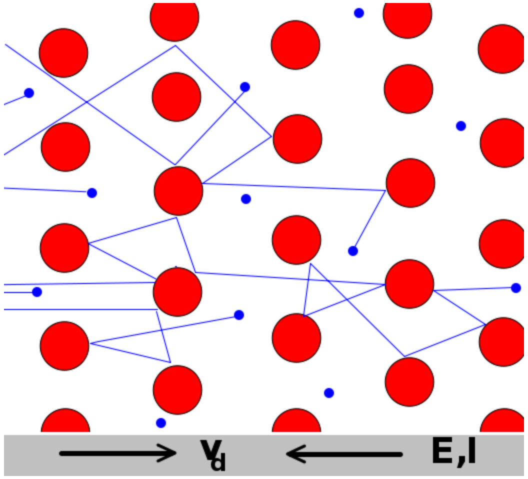
m^* - effective mass
<τ> - mean relaxation time

2.3.1. Microscopic origins of Ohm's law

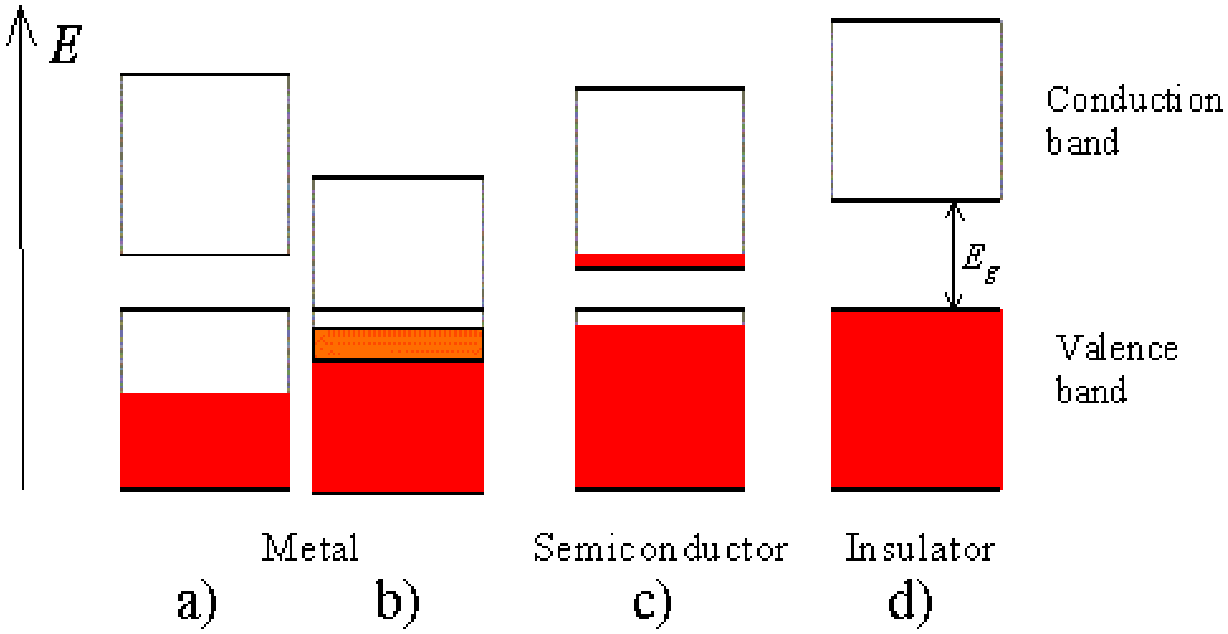
The dependence of the current density on the applied electric field is essentially quantum mechanical in nature. A qualitative description leading to Ohm's law can be based upon classical mechanics using the Drude model developed by Paul Drude in 1900.

The Drude model treats electrons (or other charge carriers) like pinballs bouncing between the ions that make up the structure of the material. Electrons will be accelerated in the opposite direction to the electric field by the average electric field at their location. With each collision, though, the electron is deflected in a random direction with a velocity that is much larger than the velocity gained by the electric field. The net result is that electrons take a tortuous path due to the collisions, **but generally drift in a direction opposing the electric field**.

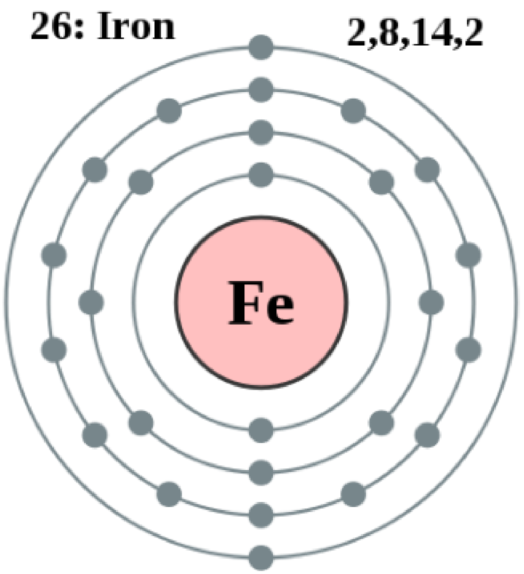
The **drift velocity** then determines the electric **current density** and its relationship to ***E*** and is independent of the collisions. Drude calculated the average drift velocity, v_d.

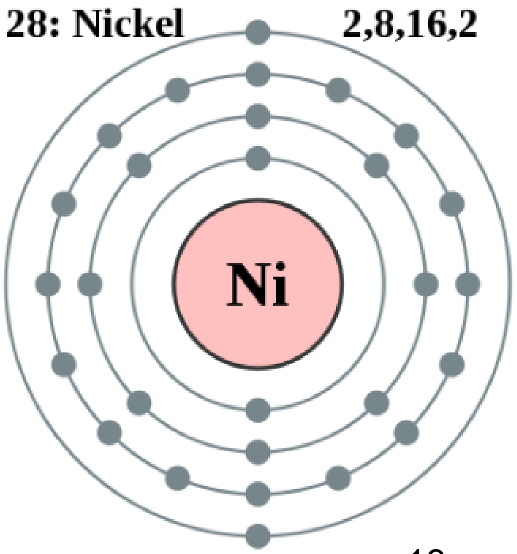


2.3.2. Classification of solids according to their conductivity



Possible energy band diagrams of a crystal. Shown are a) a half filled band (monovalent metals like Cu, Au, Ag… - high conductivity), b) two overlapping bands (ex. Fe with 2 valence electrons - 2, 8, 14, 2 -electrons/shell), c) an almost full band separated by a small bandgap, E_g , from an almost empty band and d) a full band and an empty band separated by a large bandgap





Classification of materials according to their conductivity (cont.)

(a) *Dielectrics*

- there are no free carriers (electrons)

⇩

- substances with low electric conductivity $\sigma < 10^{-5}$ S/m

(b) *Semiconductors*

$10^{-5} < \sigma < 10^{3}$ S/m

- usually $\boxed{n \sim e^{-\frac{E_g}{2K_BT}}}$ ⇨ $\boxed{\sigma = \sigma_0 \cdot e^{-\frac{E_g}{2K_BT}}}$

E_g is the *energy bandgap*

$E_g = E_c - E_v$

(c) *Conductors*

- metals
- there are electrons in C.B. – they can move almost freely in the crystalline lattice;
- $\sigma > 10^{3}$ S/m;
- Cu, Ag have $\sigma \sim 10^{7}$ S/m;

$\boxed{\rho = \rho_0\left[1 + \alpha\left(T - T_0\right)\right]}$ α is the temperature coefficient of the resistivity

ρ_0 is the resistivity at $T_0 = 273$ K

Typical electrical resistivity, ρ, and conductivity, σ, for some metals

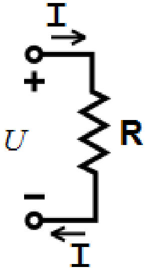
	ρ	σ
Aluminium	$2.8\ 10^{-8}\ (\Omega m)$	$3.6\ 10^{7}\ (\Omega m)^{-1}$
Copper	$1.7\ 10^{-8}\ (\Omega m)$	$5.8\ 10^{7}\ (\Omega m)^{-1}$
Iron	$1.0\ 10^{-7}\ (\Omega m)$	$1.0\ 10^{7}\ (\Omega m)^{-1}$
Silver	$1.6\ 10^{-8}\ (\Omega m)$	$6.2\ 10^{7}\ (\Omega m)^{-1}$
Nickel	$6.8\ 10^{-8}\ (\Omega m)$	$1.5\ 10^{7}\ (\Omega m)^{-1}$
Gold	$2.0\ 10^{-8}\ (\Omega m)$	$5.0\ 10^{7}\ (\Omega m)^{-1}$

2.4. Thermal effect of current - Joule heating

Joule heating, also known as resistive, resistance, or Ohmic heating, is the process by which the passage of an electric current through a conductor produces heat.
Joule heating is caused by interactions between charge carriers (usually electrons) and the body of the conductor (usually atomic ions).

The work done to transport the charge:

$$W = q \cdot U = I \cdot \Delta t \cdot U = U \cdot I \cdot \Delta t = Q \quad [\text{J}]$$



1 kWh=1000 W * 3600 s=3600 kJ

But $U=I \cdot R$ ⇨ $Q = R \cdot I^2 \Delta t$

Dissipated power in a region of circuit is:

$$P = \frac{Q}{\Delta t} = U \cdot I = I^2 R = \frac{E^2}{(R+r)^2} R \quad [\text{W}]$$

Energy dissipated throughout the circuit:

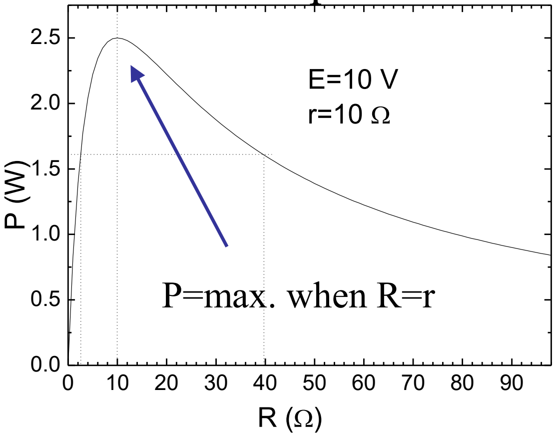
$$Q_{gen} = E \cdot I \cdot \Delta t$$

$$I = \frac{E}{R+r}$$

⇨ $Q_{gen} = \dfrac{E^2}{R+r} \Delta t \Rightarrow P_{gen} = \dfrac{E^2}{R+r} \quad [\text{W}]$

Source efficiency: $\eta = \dfrac{P}{P_{gen}} = \dfrac{\dfrac{E^2}{(R+r)^2} R}{\dfrac{E^2}{R+r}} = \dfrac{R}{r+R} = 0.5 \text{ when } r = R$

Example

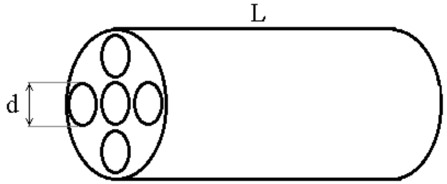


P=max. when R=r

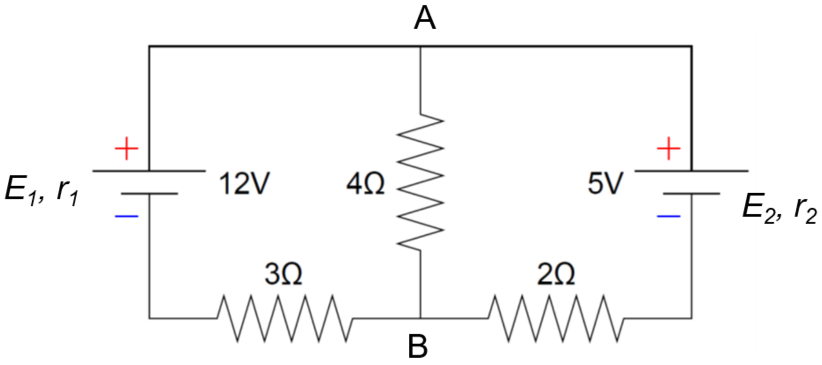
2.5. Proposed applications

1. Using Drude model, calculate the drift velocity of electrons in a Cu wire with cross section S=1 mm^2 through which is passing a current I=1 A. In 1 m^3 of copper crystal there are about 8.5×10^{28} atoms and each atom contributes with 1 electron for conduction.

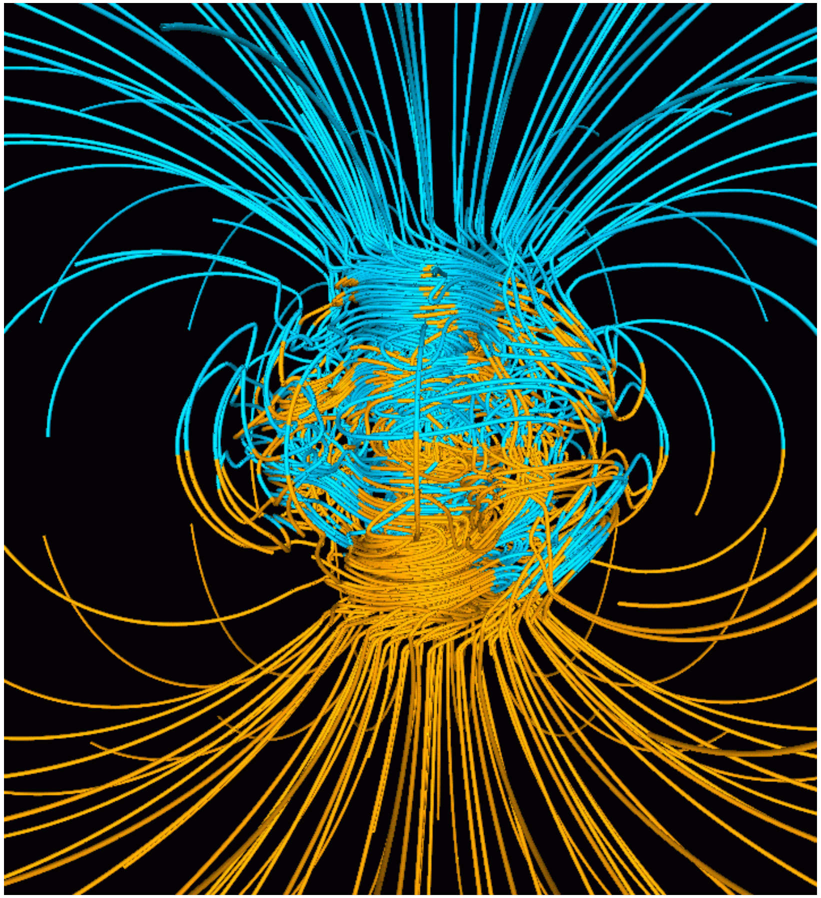
2. An electrical cable of length L=1 m contains N=5 copper wires, each with diameter d=0.2 mm. Calculate the electrical resistance of the cable. The resistivity of copper is $\rho = 1.7\cdot10^{-8}$ Ωm.



3. Determine the current through each resistor in the circuit shown below and the potential difference between A and B if:
a) $r_1=r_2=0$ and b) $r_1=2$ Ω and $r_2=1$ Ω.



Chapter III. Magnetic Field



Computer simulation of the Earth's field. From: Glatzmaier, Gary. "The Geodynamo". University of California Santa Cruz.
https://websites.pmc.ucsc.edu/~glatz/geodynamo.html

Extended Table of Contents

3.1. Introduction - Magnetic field concept

Short history (from https://en.wikipedia.org/wiki/Magnetism)

Magnetism was first discovered in the ancient world, when people noticed that lodestones, naturally magnetized pieces of the mineral magnetite, could attract iron pieces. The word magnet comes from the Greek term μαγνῆτις λίθος magnētis lithos, "the Magnesian stone, lodestone." In ancient Greece, Aristotle attributed the first of what could be called a scientific discussion of magnetism to the philosopher Thales of Miletus, who lived from about 625 BC to about 545 BC. The ancient Indian medical text Sushruta Samhita describes using magnetite to remove arrows embedded in a person's body.

Inevitability of magnetic field generation due to motion of charges

The interaction between fixed point charges is defined completely by Coulomb’s law. This law, however, *is incapable of describing the interaction between moving charges*.

The magnetic interaction is comparable to the electric interaction **only** for quit large velocities of electric particles. Nevertheless, it is also perceptible for small charge velocities if Coulomb interaction does not manifest itself for some reason or other.

Such a situation may arise, for example, when an electric current flows in a conductor. In this case, the electric field of the moving charges is neutralized by the electric field of the opposite charges of the conductor, i.e. it is screened. As a result, only the magnetic force remains.

3.2. Biot-Savart law

3.2.1. Force of interaction between parallel current-carrying conductors

Charges move in two thin cylindrical and **parallel wires**, placed in vacum, with a distance r between them. It was found that the interaction force, F_m, between them is:

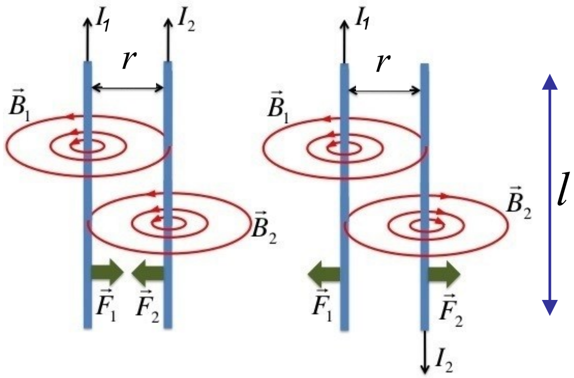


Fig. 1. Note that when the currents are parallel, an attractive force appears. When the currents are antiparallel a repulsive force appears.

$$\boxed{\begin{aligned} F_m &= \frac{\mu_0}{2\pi}\frac{I_1 I_2}{r} l \\ \vec{F}_1 + \vec{F}_2 &= 0 \\ F_1 = F_2 &= F_m \end{aligned}}$$

μ_0 - vacuum magnetic permeability

$\mu_0 = 4\pi \times 10^{-7}$ N/A $^{-2}$

$\varepsilon_0 = 8.854 \times 10^{-12}$ F/m

Between μ_0 and ε_0 is the relation: $\boxed{\mu_0 \varepsilon_0 = \frac{1}{c^2}}$

with c=299 792 458 m/s the speed of light in vacuum

For an arbitrary small length, dl, the force $dF_m = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{r} dl$ can be expressed as:

$$dF_{m,12} = \underbrace{\frac{\mu_0}{2\pi}\frac{I_1}{r}}_{B_1} \cdot I_2 \cdot dl, \quad or \quad dF_{m,21} = \underbrace{\frac{\mu_0}{2\pi}\frac{I_2}{r}}_{B_2} \cdot I_1 \cdot dl$$

Magnetic field concept where B is named magnetic flux density or simply magnetic induction

3.2.2. The magnetic field force vector

The interaction force between currents and the magnetic field can be expressed as:

$dF_m = BIdl$ B – **magnetic flux density (magnetic induction)** [T]

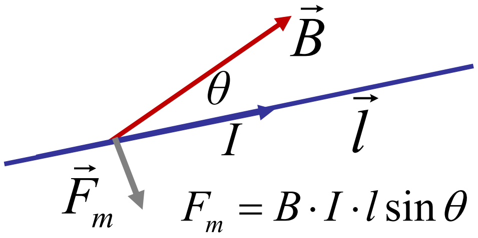
$$\boxed{\vec{F} = q\vec{E} + q \cdot \vec{v} \times \vec{B}}$$

- **generalized Lorentz force** $\vec{v}$ -the charge velocity

- in what follows, we consider only B and E=0

$$\boxed{d\vec{F}_m = Id\vec{l} \times \vec{B},\ with\ I = \frac{q}{dt},\ d\vec{l} = \vec{v}dt} \Rightarrow \boxed{d\vec{F}_m \perp d\vec{l}, \vec{B}}$$

Ampère's law



$F_m = B \cdot I \cdot l \sin\theta$

If we consider the current density, $\vec{j}$, i.e. continuous current distribution, the magnetic force is:

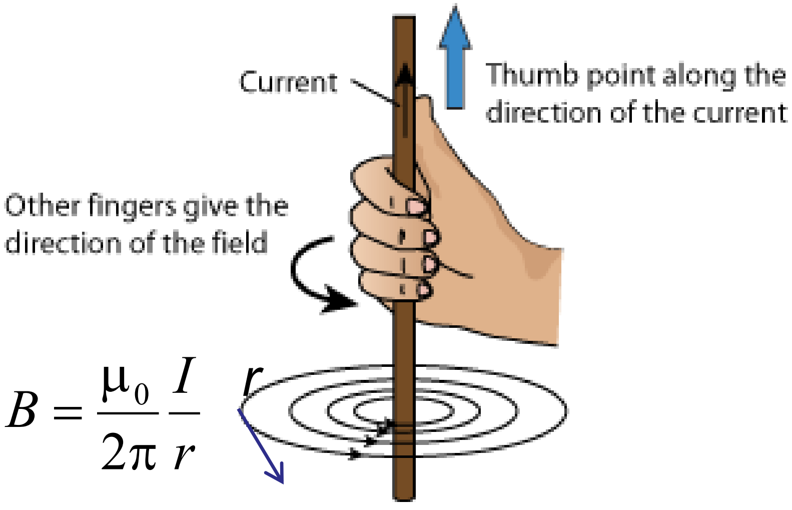
because:

$$\boxed{d\vec{F}_m = \vec{j} \times \vec{B}dV} \qquad \boxed{I = \vec{j} \cdot d\vec{S},\quad dV = d\vec{l} \cdot d\vec{S}}$$

$$\boxed{F_m = 0 \text{ if } \theta = 0, \pi}$$

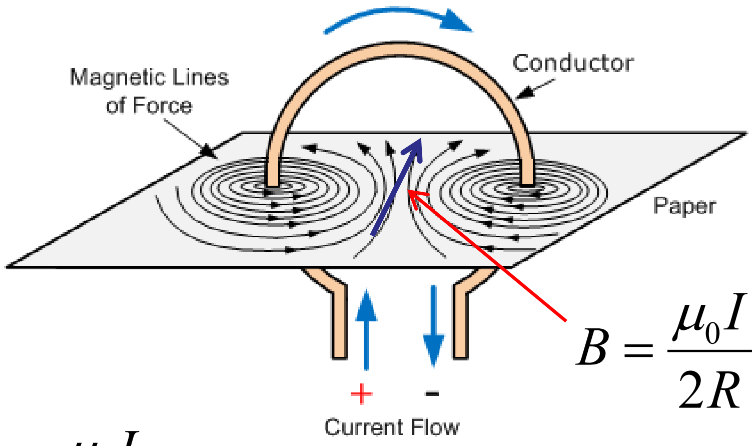
This formula reflects the basic idea of Ampère ***to reduce the interaction between current circuits to the interaction between very small current elements***

3.2.3. Field Lines around conductors



$$B = \frac{\mu_0}{2\pi}\frac{I}{r}$$

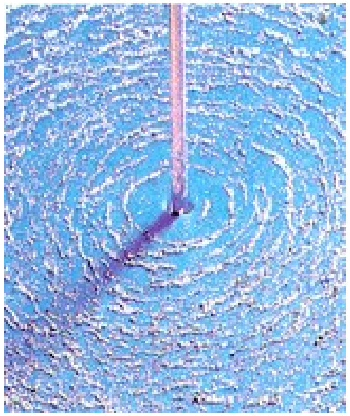
Right Hand Grip Rule



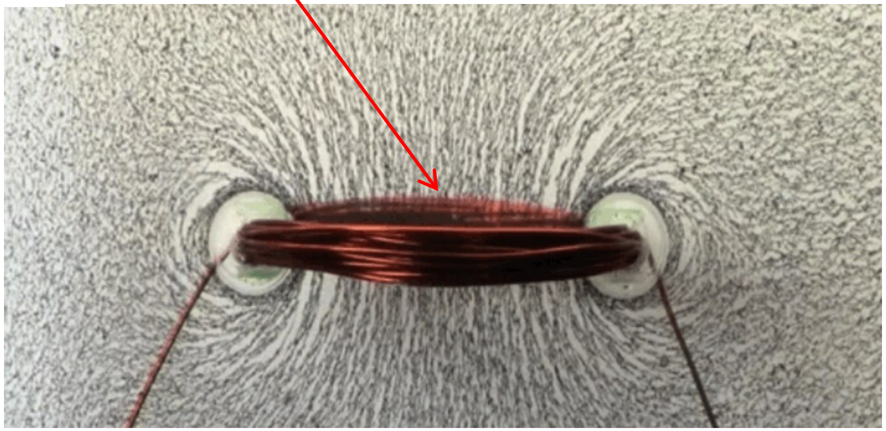
$$B = \frac{\mu_0 I}{2R}$$

$$B = N\frac{\mu_0 I}{2R}$$

N – number of turns



Magnetic field of a rectilinear current

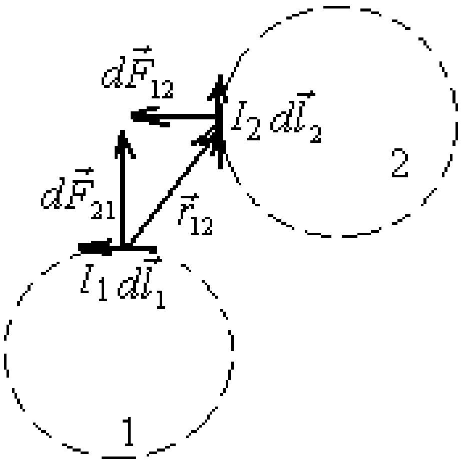


The current loop(s)

https://javalab.org/en/magnetic_field_around_a_wire_en/
https://javalab.org/en/magnetic_field_around_a_circular_wire_en/

3.2.4. The Biot-Savart law

The law of magnetic interaction between current elements was obtained in 1844 by Grassman (1809-1877). The force with which the current element $I_1 \overrightarrow{dl_1}$ acts on the current element $I_2 \overrightarrow{dl_2}$ is:



$$\boxed{d\vec{F}_{12} = \frac{\mu_0}{4\pi} \frac{I_2 d\vec{l}_2 \times (I_1 d\vec{l}_1 \times \vec{r}_{12})}{r_{12}^3}}$$

inspired from: $\boxed{d\vec{F}_{12} = I_2 d\vec{l}_2 \times d\vec{B}_{12}}$

We must note that: $d\vec{F}_{12} + d\vec{F}_{21} \neq 0$

⇩

- *the interaction of* ***current elements*** *does not obey Newton's third law*

The elementary magnetic flux density and, after in integration, the magnetic flux density:

$$\boxed{d\vec{B} = \frac{\mu_0}{4\pi} \frac{I \cdot d\vec{l} \times \vec{r}}{r^3}} \Rightarrow \boxed{\vec{B} = \frac{\mu_0}{4\pi} \int \frac{I d\vec{l} \times \vec{r}}{r^3}} \quad \boxed{\vec{B} = \frac{\mu_0}{4\pi} \int_V \frac{\vec{j} \times \vec{r}}{r^3} dV}$$

The Biot-Savart law

For a circuit with current I, or current density $\vec{j}$

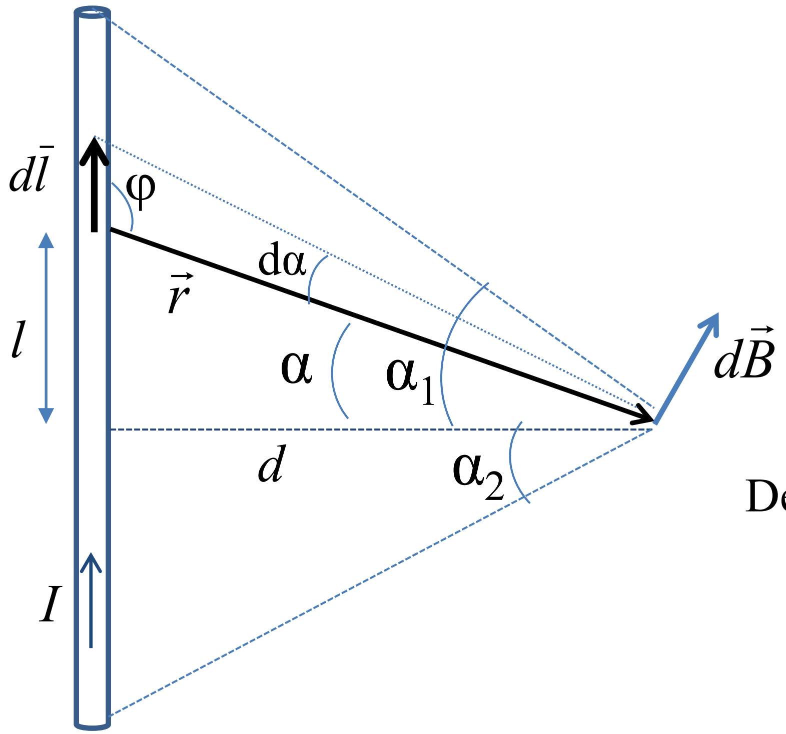
The force with which current I_1 flowing in closed loop L_1 acts on closed loop L_2 carrying current I_2, according is:

$$\boxed{\vec{F}_{12} = \frac{\mu_0 I_1 I_2}{4\pi} \int_{L_1} \int_{L_2} \frac{d\vec{l}_2 \times (d\vec{l}_1 \times \vec{r}_{12})}{r_{12}^3}}$$

In this case $\vec{F}_{12} + \vec{F}_{21} = 0$

3.2.5. Solved and proposed applications

1. Calculate the magnetic flux density of the field created by a rectilinear current *I* (discussion for the case of finite length and the infinite length conductor).

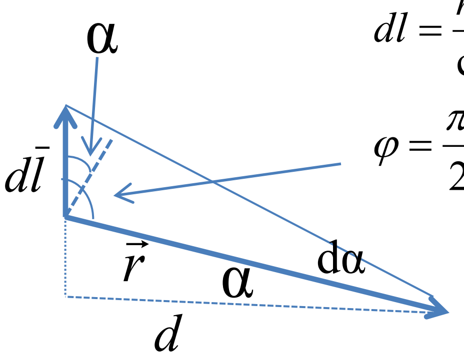


$$d\vec{B} = \frac{\mu_0 I d\vec{l} \times \vec{r}}{4\pi \cdot r^3}, \quad dB = \frac{\mu_0 I \cdot dl \cdot r \cdot \sin\varphi}{4\pi \cdot r^3}, \quad r = \frac{d}{\cos\alpha}$$

$$d\vec{B} \perp d\vec{l}, \; d\vec{B} \perp \vec{r}$$

- from basic geometry:

Detail:



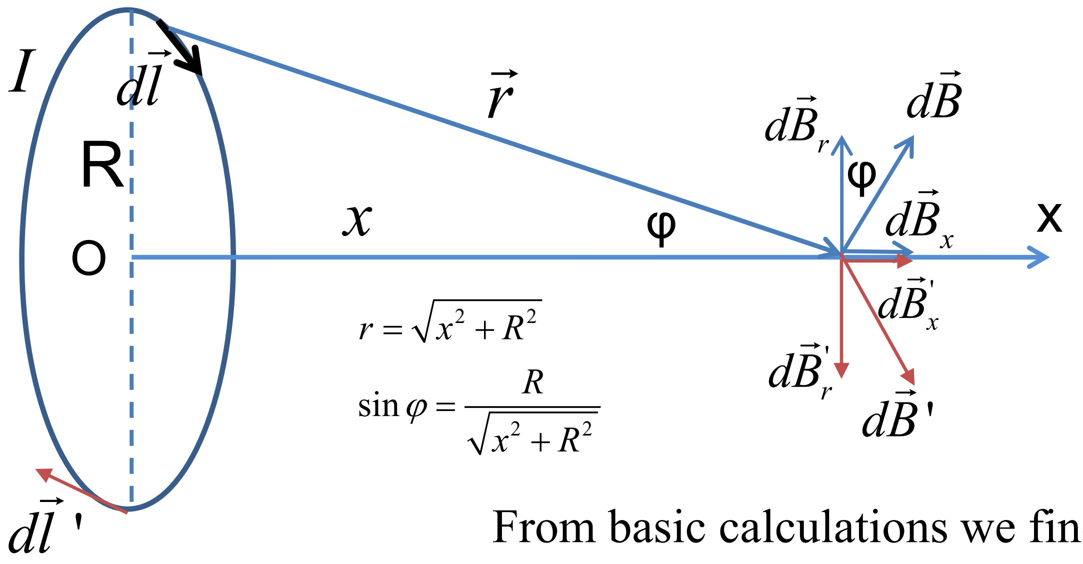
$$dl = \frac{r \cdot d\alpha}{\cos\alpha}$$

$$\varphi = \frac{\pi}{2} + \alpha$$

Using these considerations you get $B = \dfrac{\mu_0 I}{4\pi \cdot d} \displaystyle\int_{-\alpha_1}^{\alpha_2} \cos\alpha \cdot d\alpha = \dfrac{\mu_0 I}{4\pi \cdot d}\left(\sin\alpha_2 + \sin\alpha_1\right)$

For a conductor of infinite length, $\alpha_{1,2} \to \pi/2$ ➡ $B = \dfrac{\mu_0 I}{2\pi d}$

2. Using the Biot-Savart law, calculate the magnetic flux density of the field created by a circular single turn loop, of radius *R*, on his axis of symmetry, Ox, through which is flowing the current *I*. Discuss the cases: x=0 and x → ∞; What is Helmholtz coil?



$$r = \sqrt{x^2 + R^2}$$

$$\sin\varphi = \frac{R}{\sqrt{x^2 + R^2}}$$

Solving steps:

- using

$$\vec{B} = \frac{\mu_0}{4\pi}\oint\frac{Id\vec{l}\times\vec{r}}{r^3}$$

$$d\vec{B}_r + d\vec{B}'_r = 0$$

$$dB_x = dB\sin\varphi;\ dB = \frac{\mu_0}{4\pi}\frac{Idl\cdot r}{r^3}$$

From basic calculations we find: $$B_x = \frac{\mu_0 I\cdot R^2}{2\left(R^2 + x^2\right)^{3/2}} \qquad (1)$$

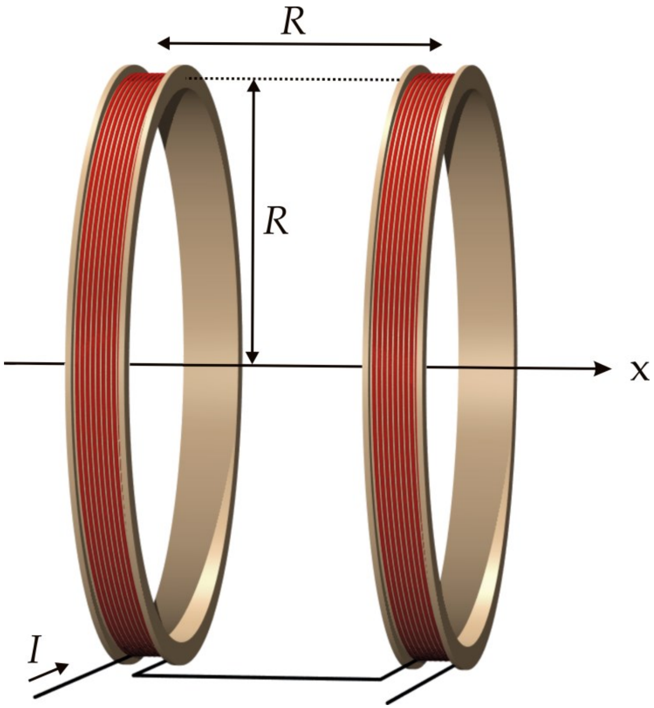
If there are N turns in the coil, then: $$B_x = N\frac{\mu_0 I\cdot R^2}{2\left(R^2 + x^2\right)^{3/2}} \qquad (2)$$

For x=0 $$B_x = N\frac{\mu_0 I}{2R}$$

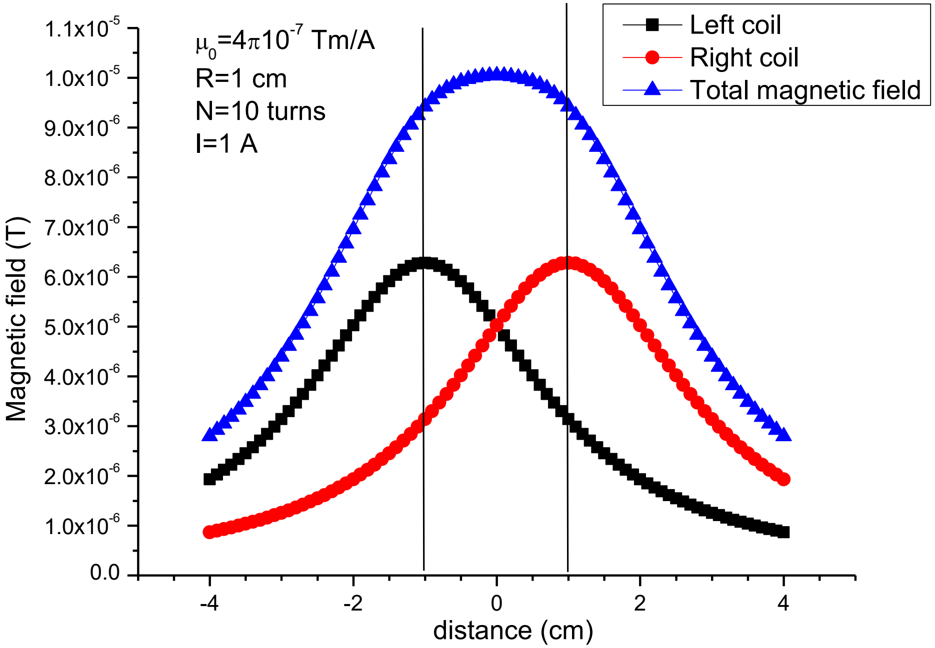
For x>>R $$B_x = N\frac{\mu_0 I\cdot\pi\cdot R^2}{2\pi\cdot x^3}\sim\frac{m}{x^3}$$

with $m = I\cdot\pi\cdot R^2$ **magnetic moment**

The Helmholtz coil consists of two identical coils with N turns each, and radius R. The distance between them is set to R. In this case, by using eq (2) and setting x=R we find:

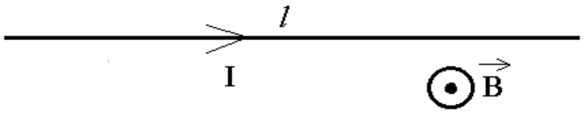


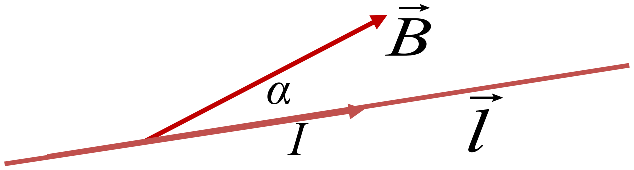
$$B = \left(\frac{4}{5}\right)^{3/2} \frac{\mu_0 N I}{R}$$



Example of magnetic field distribution inside of a Helmholtz coil

3. A horizontal linear conductor of length *l*=0.5 m carries a current *I*=10 A from left to right. The conductor is placed in a horizontal magnetic field *B*=1.5 T, perpendicular to the current, α=π/2. What is the direction and size of the force acting on the conductor? Express these results if, now, α=π/3.

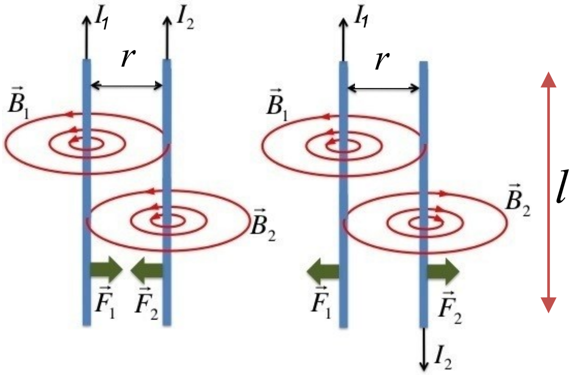




$$\vec{F}_m = I\vec{l} \times \vec{B}$$

$$F_m = B \cdot I \cdot l \sin\theta$$

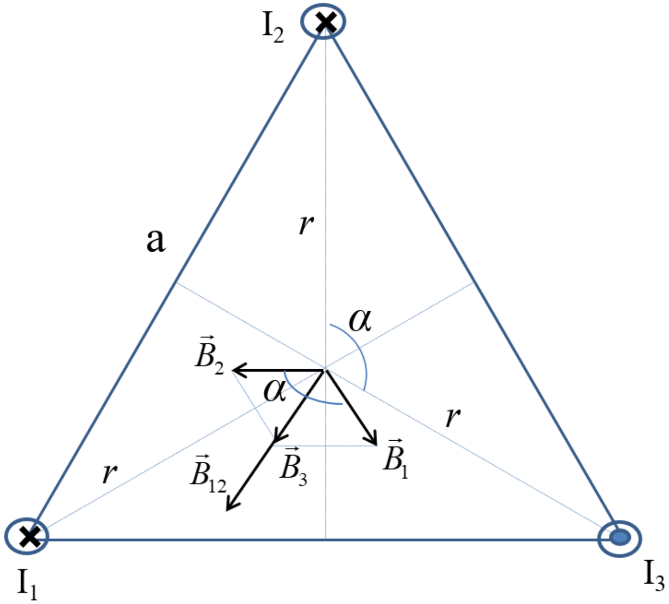
4. Two horizontal linear conductors of length *l*=0.5 m are placed in vacuum at a distance r=2 cm between them. Calculate the interaction force between conductors if the currents are I_1=10 A and I_2=20 A. How is this directed this force if a) the currents are parallel and b) antiparallel.



$$F_m = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{r} l$$

$$\mu_0 = 4\pi \times 10^{-7}\ \text{N/A}^{-2}$$

5. Three currents are flowing perpendicular to the plane with directions indicated in the figure bellow. The corners describe an equilateral triangle with the side a=10 cm; $I_1=I_2=10$ A and $I_3=-10$ A. Calculate B in the centre of triangle ($\mu_0=4\pi\cdot10^{-7}$ H/m).



$$\vec{B}=\vec{B}_1+\vec{B}_2+\vec{B}_3=\vec{B}_{12}+\vec{B}_3$$

$$\text{but }\ \vec{B}_{12}\,||\,\vec{B}_3\ \Rightarrow\ B=B_{12}+B_3$$

$$\alpha=120^\circ$$

$$\frac{a}{2}=r\cos\frac{\pi}{6}\ \Rightarrow\ r=\frac{a}{2\cos\frac{\pi}{6}}=\frac{a}{2\frac{\sqrt{3}}{2}}=\frac{a\sqrt{3}}{3}=\frac{0,1\sqrt{3}}{3}$$

$$B_1=B_2=\frac{\mu_0 I_1}{2\pi r}\ \Rightarrow\ B_{12}=\sqrt{B_1^2+B_2^2+2B_1B_2\cos\left(\frac{\pi}{2}+\frac{\pi}{6}\right)}=$$

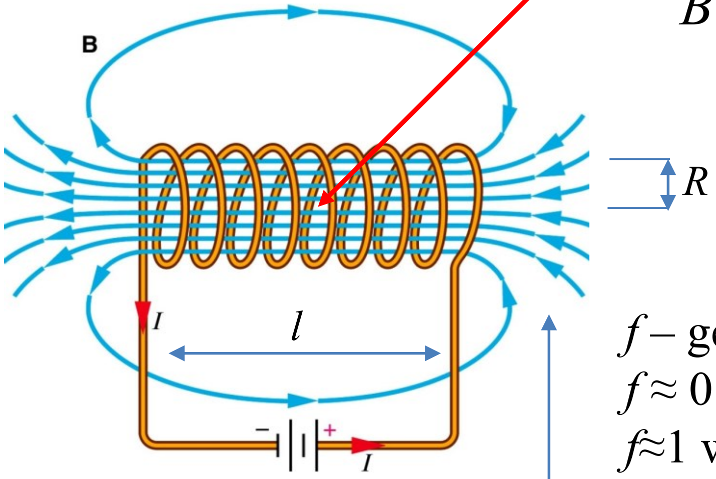
$$=\sqrt{B_1^2+B_2^2-2B_1B_2\sin\frac{\pi}{6}}=B_1\sqrt{2-2\sin\frac{\pi}{6}}=B_1$$

$$|B_3|=\frac{\mu_0|I_3|}{2\pi r}\ \Rightarrow\ B=\frac{\mu_0 I_1}{2\pi r}+\frac{\mu_0|I_3|}{2\pi r}=\frac{\mu_0}{2\pi r}\left(I_1+|I_3|\right)$$

$$B=\frac{4\pi 10^{-7}}{2\pi\cdot\frac{0,1}{\sqrt{3}}}\cdot 20=40\sqrt{3}\cdot10^{-6}=6,93\cdot10^{-5}\ \ \text{T}$$

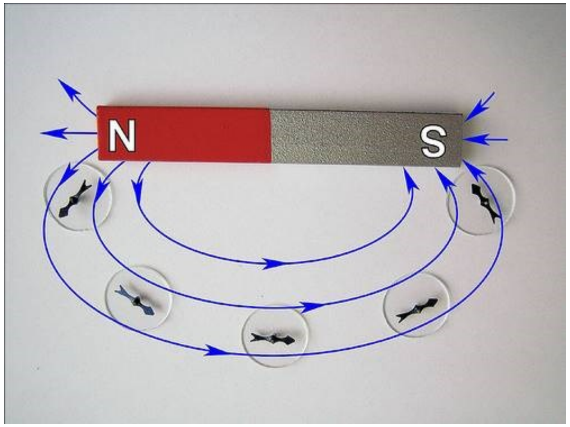
3.2.6. The solenoid

A solenoid is a coil of many circular loops wrapped up in the shape of a cylinder. We can find one big similarity between the solenoid and the bar magnet. We find that the magnetic field lines in both cases are exactly the same. In fact, both the ends of the solenoid behave as the poles. One end is the south pole and the other end is the north.
One can find that B, on the axis of symmetry and in the centre of the solenoid, is:



$$B = \frac{\mu_0 N \cdot I}{l} \cdot f$$

f – geometrical factor
$f \approx 0.58$ when l=R
$f \approx 1$ when l=5R



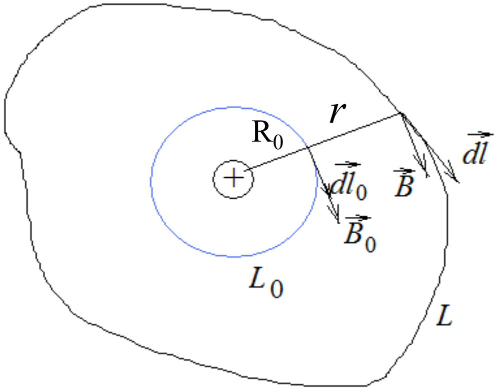
$f<1$ reflects the influence of the end effects where the magnetic field lines return.

When l>>R $B = \dfrac{\mu_0 N \cdot I}{l}$ Can you obtain this formula using the Biot-Savart law?

3.3 The Ampère’s circuital law

3.3.1. The integral form of Ampère’s circuital law

Let’s consider the field generated by the current flowing in an infinite thin very long rectilinear wire. **The circulation of the vector** $\vec{B}$ around a certain closed contour L enclosing the current I is given by:



$$\boxed{\oint_L \vec{B}d\vec{l} = \frac{\mu_0}{2\pi} I \oint_L d\alpha = \mu_0 I} \quad (1) \qquad \text{with} \quad B = \frac{\mu_0}{2\pi \cdot r} I$$

This can be derived from the following considerations:

$$\vec{B}d\vec{l} = Bdl\cos(\vec{B}, d\vec{l}) = Bdl_{\perp} = \frac{\mu_0}{2\pi} I \cdot d\alpha; \;\; dl_{\perp} = dl \cdot \cos(\vec{B}, d\vec{l}) = r \cdot d\alpha$$

Formula (1) can be derived, also, from:

$$\oint_{L_0} \vec{B}_0 d\vec{l}_0 = \frac{\mu_0 I}{2\pi R_0} R_0 \oint_L d\alpha = \mu_0 I \qquad \text{but} \quad \oint_{L_0} \vec{B}_0 d\vec{l}_0 = \oint_L \vec{B}d\vec{l} = \mu_0 I$$

If a closed contour L’ does not embrace a net current, *I*, we get $\oint_{L'} \vec{B}d\vec{l} = 0$

Suppose that we have a large number of currents

$$\vec{B} = \sum_i \vec{B}_i \qquad \boxed{\oint_L \vec{B}d\vec{l} = \sum_k \mu_0 I_k = \mu_0 I}$$

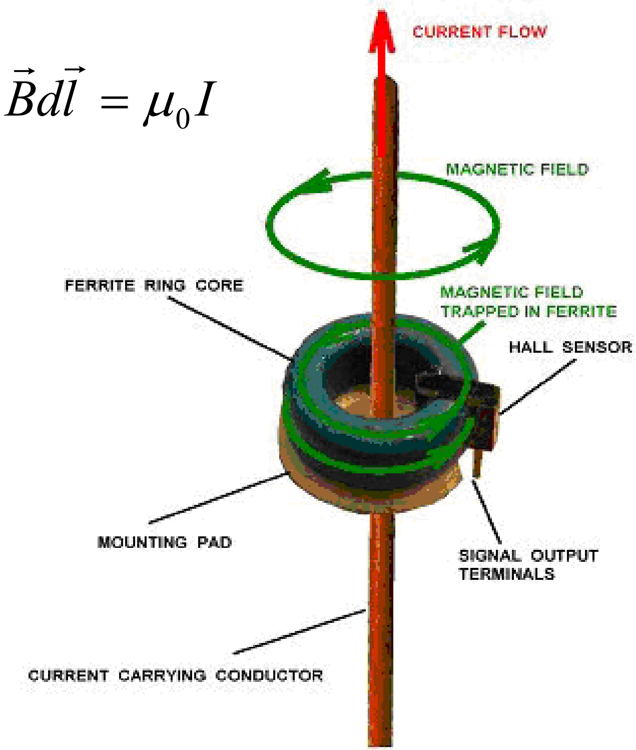
The integral form of Ampere’s circuital law

I is the net current embraced by the contour L

-the subscript k denotes only the currents embraced by the contour L

The Ampère's Circuital Law has many applications in magnetic circuit design and detection of the electric current through a conductor using Hall sensors which are used to measure the circulation of $\vec{B}$ over a closed contour defined by a magnetic core.

$$\oint_L \vec{B}d\vec{l} = \mu_0 I$$

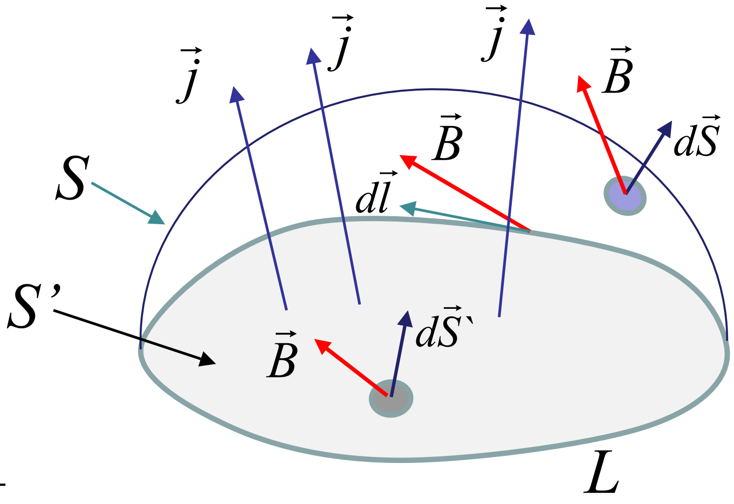


Hall effect current transducer (galvanic isolator).

clamp-meter

3.3.2. Differential form of Ampère's law

- for volume currents $I = \int_S \vec{j}d\vec{S}$ or $I = \int_{S'} \vec{j}d\vec{S}`$



The following steps are obvious:

$$\oint_L \vec{B}d\vec{l} = \mu_0 I \Longrightarrow \oint_L \vec{B}d\vec{l} = \mu_0 \iint_S \vec{j}d\vec{S}$$

$$\oint_L \vec{B}d\vec{l} = \iint_S rot\vec{B}\,d\vec{S}$$ - from Stokes's theorem

S is the surface surrounded by the closed contour *L*

$$\Downarrow$$

$$\iint_S \left[rot\vec{B} - \mu_0\vec{j}\right]d\vec{S} = 0 \Longrightarrow \boxed{curl\vec{B} \equiv rot\vec{B} = \nabla \times \vec{B} = \mu_0\vec{j}}$$

Maxwell's Equations for a Stationary Magnetic Field

$$\boxed{div\vec{B} = 0} \qquad \boxed{rot\vec{B} \equiv \nabla \times \vec{B} = \mu_0\vec{j}}$$

From the equation $div\vec{B} = 0$ we conclude that ***the lines of B do not have sources***.

This means that ***there are no magnetic charges which would generate a magnetic field*** in the same way as electric charges create an electric field. The lines of $\vec{B}$ have neither beginning nor end. They are either closed or go to infinity.

3.3.3. Magnetic vector potential approach

The identity $div\left(rot\right)\equiv 0$ shows that the solution of equation $div\vec{B}=0$

can be expressed in the form $\boxed{\vec{B}=curl\vec{A}\equiv rot\vec{A}\equiv\nabla\times\vec{A}}$

$\vec{A}$ is named the **vector potential** of a magnetic field

The vector potential, $\vec{A}$, plays only an auxiliary role and cannot be measured experimentally like the electric potential. $\vec{A}$ is mainly used for calculation purposes, especially in quantum mechanics.

In magnetostatics $\boxed{div\vec{A}=0}$ **- represents the gauging condition for the potential**

Because $rot\,\vec{B}=\mu_0\vec{j}$ we have $rot\,(rot\,\vec{A})=\mu_0\vec{j}$ (1)

From $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\vec{b}\left(\vec{a}\cdot\vec{c}\right)-\vec{c}\left(\vec{a}\cdot\vec{b}\right)$

$$\nabla\times\left(\nabla\times\vec{f}\right)=\nabla\left(\nabla\cdot\vec{f}\right)-\left(\nabla\cdot\nabla\right)\vec{f}=\nabla\left(\nabla\cdot\vec{f}\right)-\nabla^2\vec{f}$$

⇓

$$curl\ curl\vec{A}=grad\ div\vec{A}-\nabla^2\vec{A}\quad\Rightarrow\quad\boxed{\nabla^2\vec{A}=-\mu_0\vec{j}}$$

Because of the gauge condition and eq. (1)

In terms of x, y, z coordinates we can write

$$\nabla^2 A_x = -\mu_0 j_x \qquad \nabla^2 A_y = -\mu_0 j_y \qquad \nabla^2 A_z = -\mu_0 j_z$$

Each component of the vector potential satisfies Poisson's equation. If all currents are concentrated in a finite region of space then we can write the solution in the form:

$$A_x = \frac{\mu_0}{4\pi}\int \frac{j_x dV}{r} \qquad A_y = \frac{\mu_0}{4\pi}\int \frac{j_y dV}{r} \qquad A_z = \frac{\mu_0}{4\pi}\int \frac{j_z dV}{r}$$

or in the vector form :

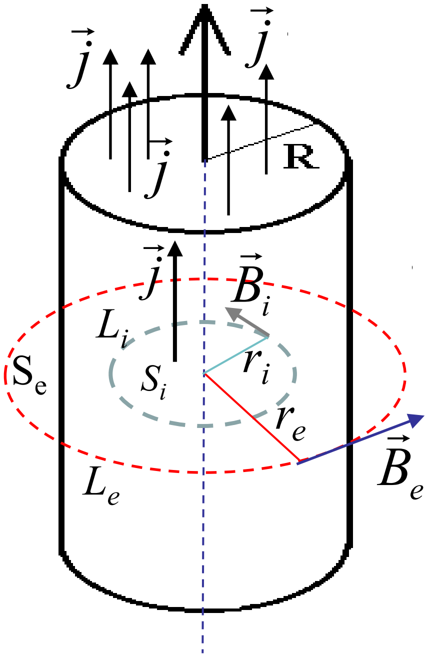
$$\boxed{\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{j}dV}{r}} \qquad \boxed{\vec{B} = \frac{\mu_0}{4\pi}\int_V \frac{\vec{j}\times\vec{r}}{r^3}dV}$$

<u>For a line current these equations become :</u>

$$\boxed{\vec{A} = \frac{\mu_0}{4\pi}\int_L \frac{Id\vec{l}}{r} = \frac{\mu_0}{4\pi}\sum_i I_i \int_{L_i} \frac{d\vec{l}}{r}} \qquad \boxed{\vec{B} = \frac{\mu_0}{4\pi}\int \frac{Id\vec{l}\times\vec{r}}{r^3}}$$

3.3.4. Applications using the Ampère’s circuital law

1. A cylindrical conductor of infinite length and radius R carries a current, I, with uniform current density j ($I=j\cdot\pi\cdot R^2$). Calculate the magnetic field as a function of radius r from the center of the conductor ($r \leq R$, $r > R$). Consider $I=20$ A and $R=2$ mm.



a) $r_i \leq R$

Ampere’s circuital law applied to contour L_i is: $\oint_{L_i} \vec{B}_i d\vec{l}_i = \mu_0 I_i$

- where I_i represents the current embraced by L_i i.e. the current through surface S_i.

But: $I_i = \int_{S_i} \vec{j} d\vec{S}_i = j \cdot \pi r^2$ because j is uniform un S

$\vec{B}_i d\vec{l}_i = B_i dl_i$ because $B_i \perp r_i \Rightarrow \vec{B}_i \parallel d\vec{l}_i$ on any point on L_i

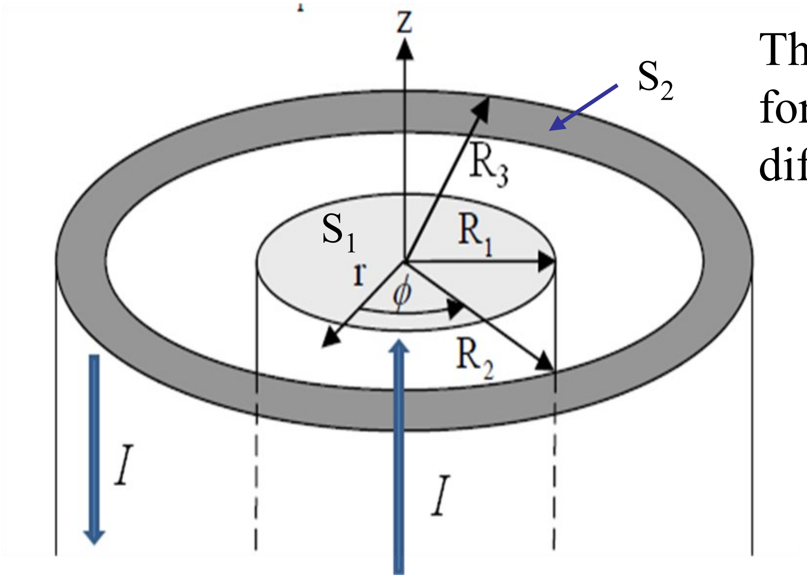
$$\oint_{L_i} \vec{B}_i d\vec{l}_i = \mu_0 I_i \Rightarrow B_i \cdot 2\pi r_i = \mu_0 j \cdot \pi r_i^2 \Rightarrow B_i = \frac{\mu_0 j}{2} r_i$$

But $I=j\cdot\pi\cdot R^2$ ⇨ $B_i = \frac{\mu_0 j}{2} r_i = \frac{\mu_0 I}{2\pi R^2} r$

b) $r_e > R$ $\oint_{L_e} \vec{B}_e d\vec{l}_e = \mu_0 I$ because current embraced by L_e is I.

From: $B_e \cdot 2\pi r_e = \mu_0 I \Rightarrow B_e = \frac{\mu_0 I}{2\pi r_e}$

2. A coaxial cable of infinite length is composed of an inner cylindrical conductor of radius R_1 and an outer cylindrical shell conductor between radii R_2 and R_3. Between them there is vacuum. The cable carries a current I with uniform current density through each conductor. Using the Ampere's circuital law calculate the magnetic field as a function of radius r.



The solving steps must follow the same approach like for problem 1; however, the current densities are different, i.e. $j_1 = I/S_1 = I/(\pi R_1^2)$ and $j_2 = I/S_2 = I/(\pi R_3^2 - \pi R_2^2)$

a) $r \le R_1$

$$B_1 = \frac{\mu_0 j_1}{2} r = \frac{\mu_0 I}{2\pi R_1^2} r$$ - see problem 1

b) $R_1 < r < R_2$ - the current embraced is I

$$B_2 \cdot 2\pi r = \mu_0 I \Rightarrow B_2 = \frac{\mu_0 I}{2\pi r}$$ - see problem 1

c) $R_2 \le r \le R_3$ $\oint_{L_3} \vec{B}_3 d\vec{l}_3 = \mu_0 I_{net}$ where $I_{net} = I - j_2\left(\pi r^2 - \pi R_2^2\right)$

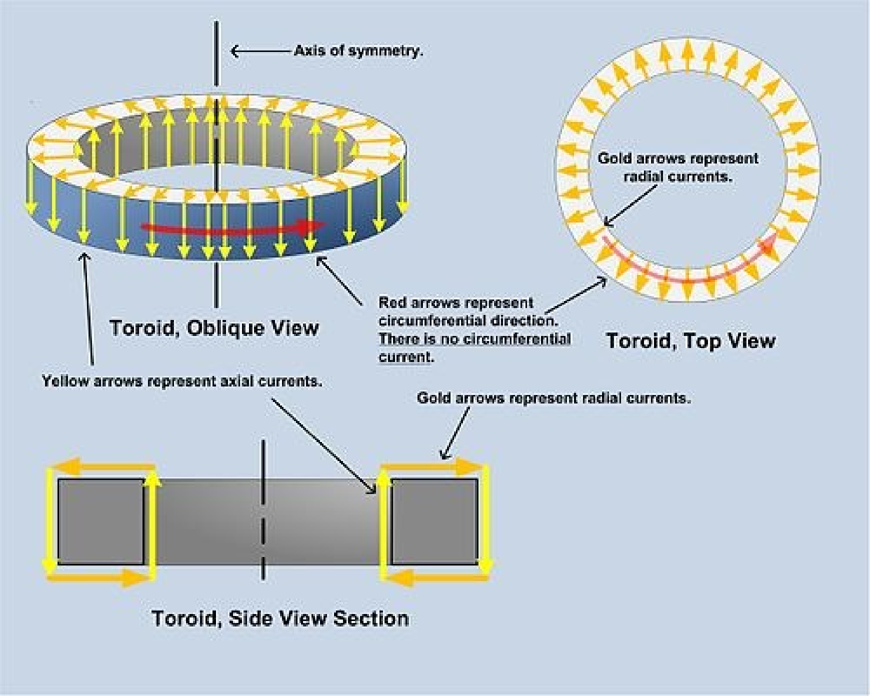
- following the same calculation steps: $B_3 = \frac{\mu_0 I}{2\pi r} - \frac{\mu_0 I}{2\pi r} \frac{\pi\left(r^2 - R_2^2\right)}{\pi\left(R_3^2 - R_2^2\right)} = \frac{\mu_0 I}{2\pi r}\left(1 - \frac{r^2 - R_2^2}{R_3^2 - R_2^2}\right)$

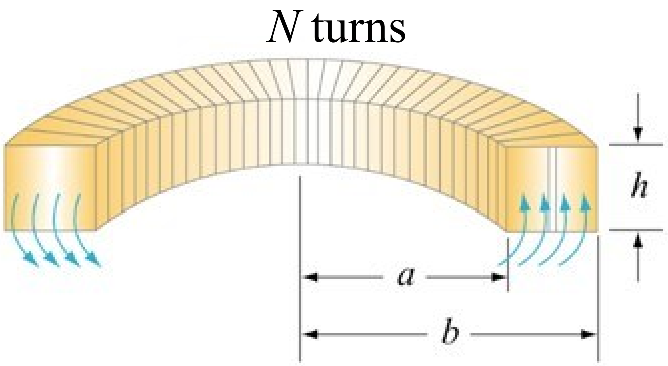
d) $r > R_3$ $\oint_{L_4} \vec{B}_4 d\vec{l}_4 = \mu_0 I_{net}$ where $I_{net} = 0 \Rightarrow B_4 = 0$

Consider I=10 A, R_1=1 mm, R_2=3mm and R_3=4 mm. Can you plot $B(r)$ from problems 1 and 2?

3. Determine the magnetic flux density (magnetic induction), along the radius r, generated by a toroidal coil that has an inner radius R_i=a=3 cm, the external radius R_e=b=5 cm and a height h=1 cm (rectangular cross section). The coil has N=200 turns of wire. What is the total magnetic flux generated inside of this toroidal coil? **Hint**: Use Ampere's circuital law.

A toroidal coil looks like in the figures bellow.





The solving steps will follow the same approach like for problem 2, where different regions must be considered.

a) $r < R_i$

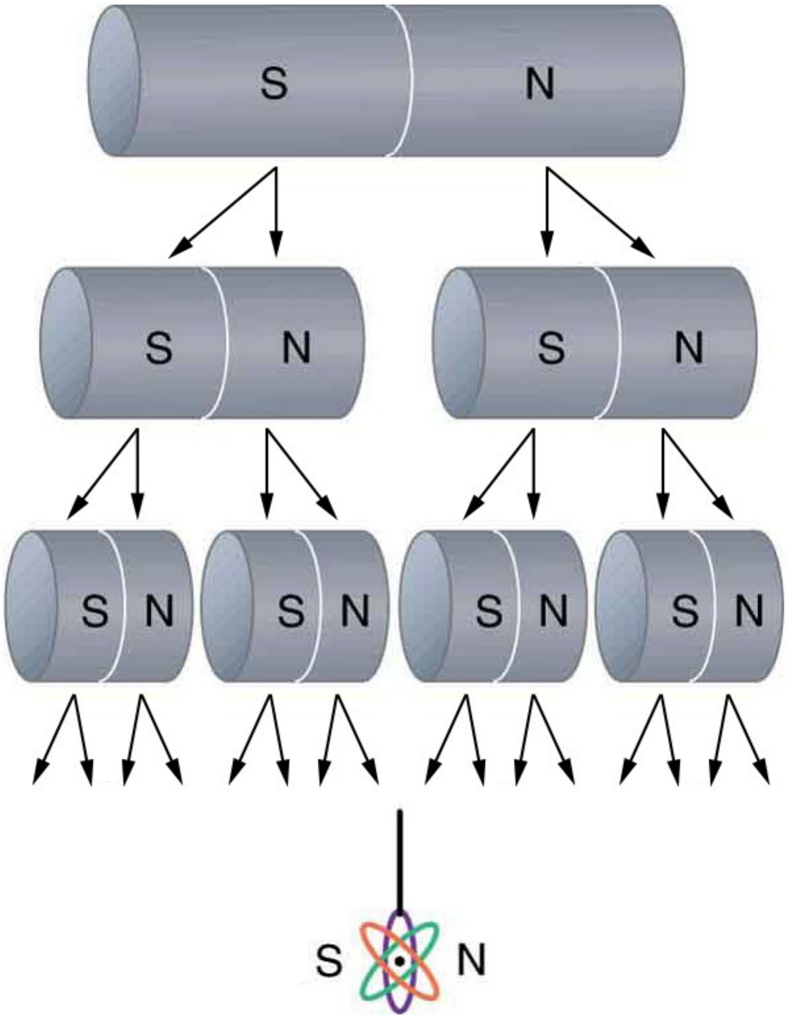
$$\oint_{L_1} \vec{B}_1 d\vec{l}_1 = \mu_0 I_{net} \quad \text{where } I_{net} = 0 \Rightarrow B_1 = 0$$

- no current is enclosed by the contour L_1 with $r<R_1$

b) $R_i \le r < R_e$ $\quad \oint_{L_2} \vec{B}_2 d\vec{l}_2 = \mu_0 I_{net} \quad \text{where } I_{net} = N \cdot I \Rightarrow B_2(r) = \dfrac{\mu_0 N \cdot I}{2\pi r}$

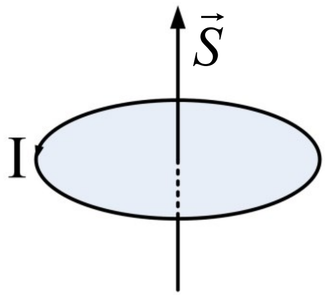
c) $r \ge R_3$ ➔ I_{net}=0 ➔ B_3=0 Can you plot $B(r)$?

3.4. Magnetic field in the presence of magnetics



3.4.1. The field of an elementary dipole

Consider a current line which is flowing around a surface with very small linear dimensions.



We can define a quantity named “Magnetic dipole moment” of the elementary current by:

$$\boxed{\vec{m} = I\vec{S}}$$

By analogy with the electric field produced by an electric dipole:

$$\vec{E} = -grad\varphi = \frac{1}{4\pi\varepsilon_0}\left[\frac{3(\vec{p}\vec{r})\vec{r}}{r^5} - \frac{\vec{p}}{r^3}\right]$$

one can write:

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi}\left[\frac{3\left(\vec{m}\cdot\vec{r}\right)\vec{r}}{r^5} - \frac{\vec{m}}{r^3}\right]}$$

The magnetic field corresponding to the magnetic moment decreases in inverse proportion to the third power of the distance.

From a previous calculation, magnetic field along the axis of a circular coil carrying a current I, was found to be:

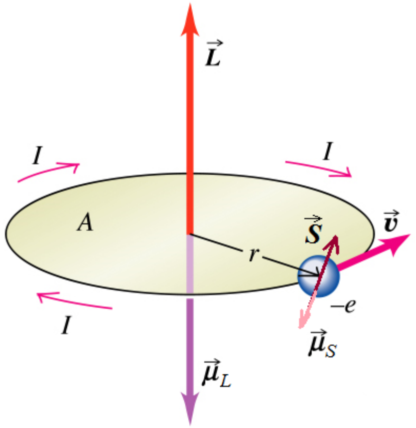
$$B_x = \frac{\mu_0 I\cdot\pi\cdot R^2}{2\pi\cdot x^3} \sim \frac{m}{x^3}\ , x \gg R$$

with $m = I\cdot\pi R^2$

3.4.2. Mechanisms of magnetization

Magnetics are substances which, upon being introduced into an external magnetic field, change so that they themselves become sources of an additional magnetic field.
The total magnetic induction in this case is the sum of the inductions of the external magnetic field and the magnetic field generated by the substance.

The change in state of a substance under the action of an external magnetic field is called ***magnetization***.



The physical cause of the magnetism of objects, as distinct from electrical currents, is the atomic magnetic dipole. Magnetic dipoles, or magnetic moments, result on the atomic scale from the two kinds of movement of electrons:

- orbital motion of the electrons around the nucleus, $\vec{L}$; this motion can be considered as a current loop, I, resulting in an orbital dipole magnetic moment $\vec{\mu}_L$ along the axis of the nucleus;
- spinning of electrons; because of this we say that the electron poses a spin moment, $\vec{S}$, and a spin magnetic moment, $\vec{\mu}_S$. This is a quantum mechanical property called the spin dipole magnetic moment (although current quantum mechanical theory states that electrons neither physically spin, nor orbit the nucleus). This behaviour gives us the image of an electron which is ***spinning***. Because of this we say that the electron poses a ***spin moment*** and a ***spin magnetic moment***.

The overall magnetic moment of the atom is the net sum of all of the magnetic moments of the individual electrons. Because of the tendency of magnetic dipoles to oppose each other to reduce the net energy, in an atom the opposing magnetic moments of some pairs of electrons cancel each other, both in orbital motion and in spin magnetic moments. Thus, in the case of an atom with a completely filled electron shell or subshell, the magnetic moments normally completely cancel each other out and only atoms with partially-filled electron shells have a magnetic moment, whose strength depends on the number of unpaired electrons.

The differences in configuration of the electrons in various elements thus determine the nature and magnitude of the atomic magnetic moments, which in turn determine the differing magnetic properties of various materials. Several forms of magnetic behaviour have been observed in different materials, including:

- **Diamagnetism**
- **Paramagnetism**
- **Ferromagnetism**
- **Antiferromagnetism**
- **Ferrimagnetism**

Mechanisms of magnetization

1. The molecules of the substance introduced into the magnetic field *acquire an induced magnetic moment – they become the sources of an additional field.* Such substances are called **diamagnetics**.
2. The molecules can have a magnetic moment even in the absence of magnetic field, i.e. the ***molecules possess a permanent magnetic moment*** – each molecule is a source of magnetic field. In the absence of an external magnetic field, the magnetic moments of different molecules are oriented quite randomly so that the total magnetic induction created by the system is zero – the body is not magnetized. When an external magnetic field is applied, the permanent magnetic moments of individual molecules are reoriented in the direction of the external magnetic induction. The substance is magnetized. Such materials are called **paramagnetics**.
3. Magnetization of **ferro- and ferrimagnetics** - the electrons have a magnetic moment which is in a certain relation with their intrinsic angular momentum, viz., the spin. Magnetization of this class of magnetics is associated with a certain orientation of spins and is called the spin magnetization. In a few materials, the ferromagnetic ones, the magnetic dipoles tend to align, spontaneously, in the same direction because of a quantum mechanical effect named the exchange interaction.

3.4.3. Magnetization and magnetic field strength

$$\boxed{\vec{M} = \frac{1}{\Delta V}\sum_i \vec{m}_i \quad [\mathrm{A/m}]}$$

- represents the density of magnetic dipole moments in a magnetic material

- see analogy with dielectric polarization; ΔV – material volume

$\vec{m}_i$ - elementary magnetic moment

Magnetic field strength

In the absence of magnetics: $\nabla \times \vec{B} = \mu_0 \vec{j}$ $\quad\vec{j}$ - conduction current density

In the presence of magnetics:

$$\nabla \times \vec{B} = \mu_0\left(\vec{j} + \vec{j}_m\right) = \mu_0\left(\vec{j} + \nabla \times \vec{M}\right)$$

with $\vec{j}_m = \nabla \times \vec{M}$ - molecular currents density

$$\boxed{\nabla \times \left(\vec{B}/\mu_0 - \vec{M}\right) = \vec{j}} \Rightarrow \boxed{\nabla \times \vec{H} = \vec{j}} \quad \text{and} \quad \boxed{\oint_L \vec{H}d\vec{l} = I}$$

the Ampere’s circuital law for ***H***

with $\boxed{\vec{H} = \vec{B}/\mu_0 - \vec{M} \quad [\mathrm{A/m}]}$

the magnetic field strength

3.4.4. The relation between $\vec{M}$ and $\vec{H}$

To quantify the material’s magnetization in magnetic field, can be used the equation:

$$\boxed{\vec{M} = \chi \vec{H}}$$

-where χ is the magnetic susceptibility (for isotropic materials)

$$\boxed{\vec{B} = \mu_0 \vec{H} + \mu_0 \vec{M} = \mu_0 (1 + \chi) \vec{H} = \mu \vec{H}}$$

$$\boxed{\mu = \mu_0 (1 + \chi)}$$

- μ is the magnetic permeability of the medium

- μ and χ does not depend on *H* for dia- and paramagnetics

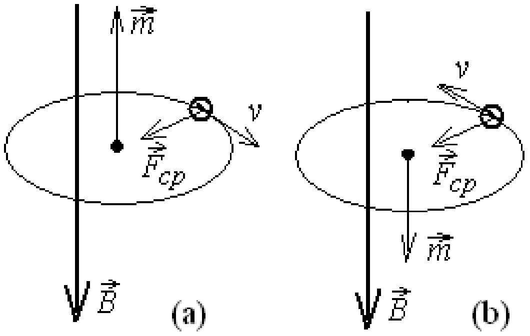
$$\boxed{\mu_r = \mu / \mu_0 = 1 + \chi}$$

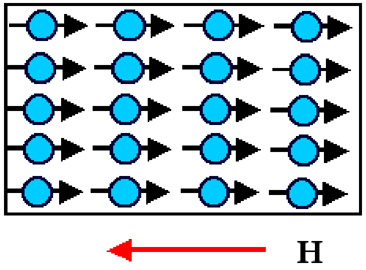
is the *relative magnetic permeability* of the medium

The magnetic susceptibility, χ, depends on the nature of substance, i.e. on the mechanism of magnetization

Diamagnetism

- Currents (and small magnetic moments) are induced by turning on the field, *B*, because the orbiting electrons are slightly disturbed and, as a result: a) increases its moment out of the page and b) decreases its moment into the page. So, the induced magnetic moments oppose the field.
- This effect shows no temperature dependence.





- all elements with filled shells (always even atomic number)
- all noble gases, H_2, NaCl, ...

Alkali or halogene *ions*

Water, DNA, most organic compounds such as oil and plastic, and many **metals** such as Cu, Pb (lead), Na, Ag, mercury, gold and bismuth show diamagnetic behaviour.

Some extreme values of χ_{dia} are given below:

$$\chi_{dia} = -9.05 \cdot 10^{-6}$$ **water**

$$\chi_{dia} = -166 \cdot 10^{-6}$$ **bismuth**

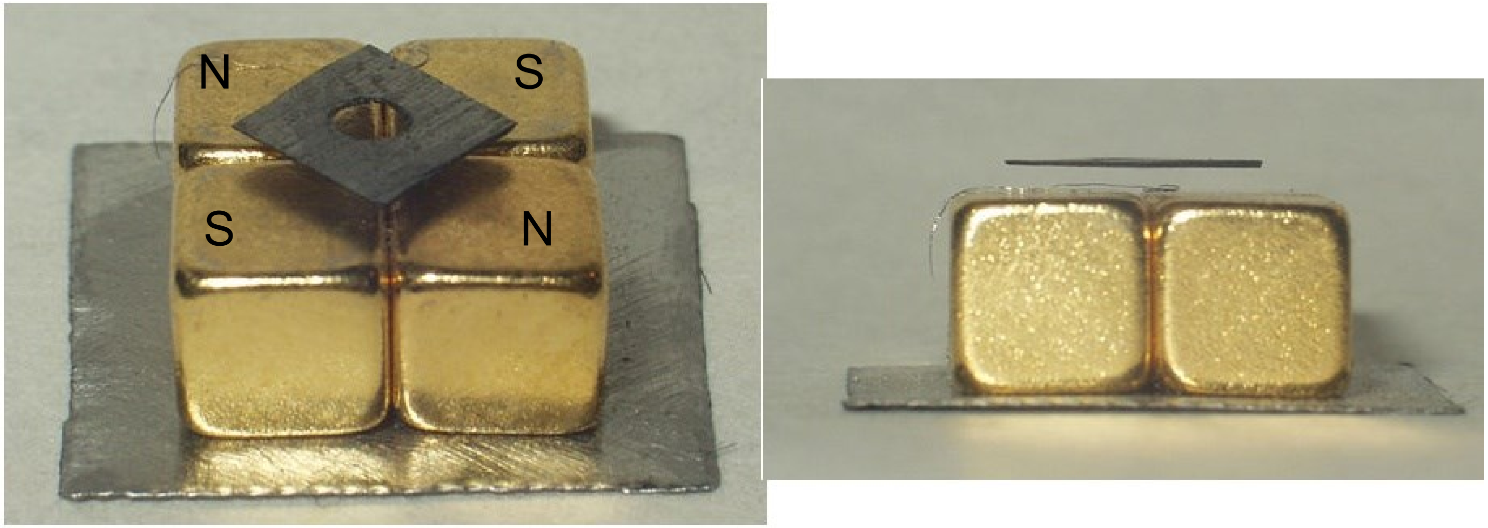
$$\chi_{dia} = -10.0 \cdot 10^{-6}$$ **Cu**

$$\chi_{dia} = -400 \cdot 10^{-6}$$ **pyrolytic graphite**

Superconductors may be considered to be perfect diamagnets ($\chi_{dia} = -1$), since they expel all field from their interior due to the Meissner effect.

Some interesting consequences: magnetic levitation

A live frog levitates inside a 32 mm diameter vertical bore of a Bitter solenoid in a magnetic field of about 16 T at the High Field Magnet Laboratory of the Radboud University in Nijmegen the Netherlands



A small (~6 mm) piece of pyrolytic graphite levitating over a permanent neodymium magnet array (5mm cubes on a piece of steel). Note that the poles of the magnets are aligned vertically and alternate (two with north facing up, and two with south facing up, diagonally)

Paramagnetism

Constituent atoms or molecules of paramagnetic materials have permanent magnetic moments (dipoles), *even in the absence of an applied field*.

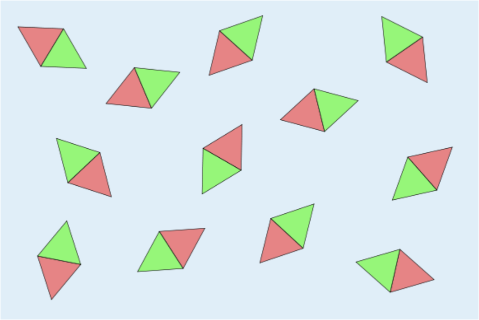
- the magnetic susceptibility is of the order of 10^{-3} to 10^{-5}

$$M = \chi H = C \cdot \frac{H}{T}$$ **Curie's Law**

a) Elements: Al, Ba, Ca, O, Pt, Na, Sr, U, Mg, Tc (Technetium – artificial), Dy, W

b) Compounds - many salts of the *d* and *f* transitional metal group show paramagnetic behaviour. Some of them are: Copper sulphate ($CuSO_4$), Ferric chloride ($FeCl_3$), Manganese chloride ($MnCl_2$)

Material	χ ($*10^{-5}$)
Tungsten (W)	6.8
Caesium (Cs)	5.1
Aluminium (Al)	2.2
Lithium (Li)	1.4
Magnesium (Mg)	1.2
Sodium (Na)	0.72



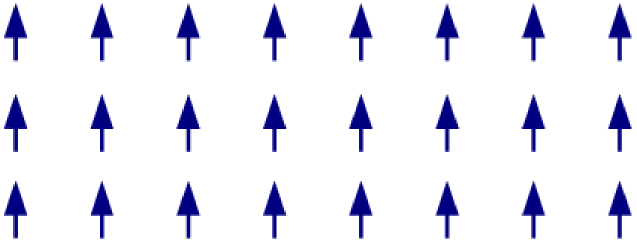
Random orientation of magnetic moments when the external field is 0

Ferromagnetism

The term "**ferromagnet**" was used for any material like Fe, Ni, Co,..., or alloys of these materials that could exhibit *spontaneous* magnetization i.e. a net magnetic moment in the absence of an external magnetic field.

The spin of the electrons in atoms is the main source of ferromagnetism, although there is also some contribution from the orbital angular momentum of the electron around the nucleus, whose classical analogy is a current loop.

- classical electromagnetism: two nearby magnetic dipoles will tend to align in opposite directions.
- in a few materials, the ferromagnetic ones, the magnetic dipoles tend to align, spontaneously, in the same direction because of a quantum mechanical effect: **the exchange interaction**.



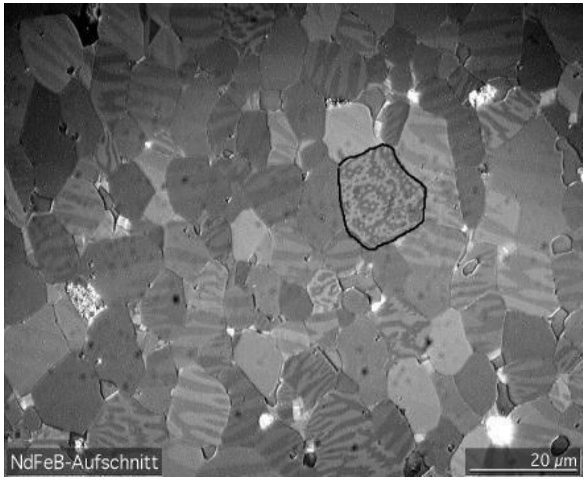
Ferromagnetic ordering of microscopic magnets

every piece of ferromagnetic material should be naturaly magnetized?

No!

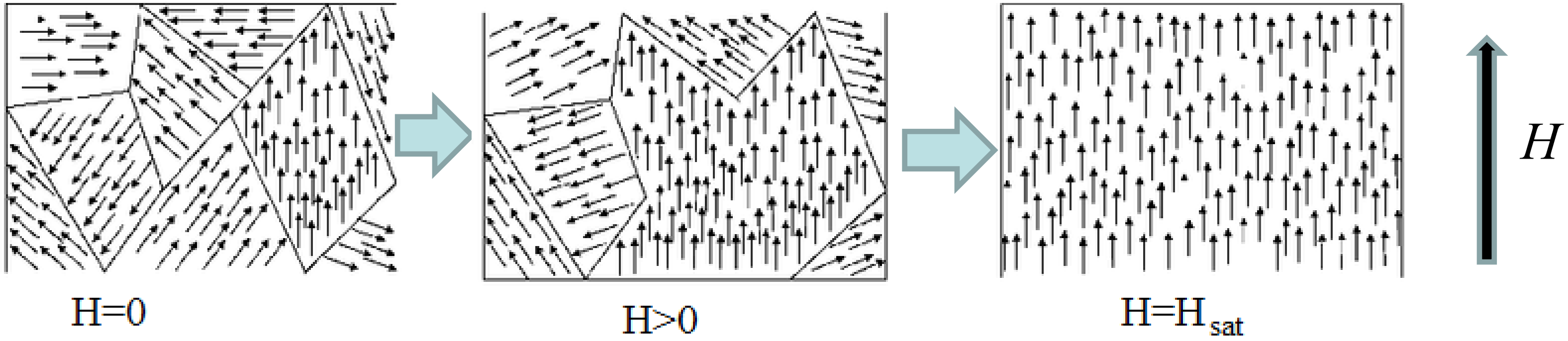
Magnetic domains

A cast piece of ferromagnetic material is divided into many tiny **magnetic domains** (also known as Weiss domains) with random orientation of their magnetic moments such that his free magnetic energy is minimized. So, the material has 0 net magnetic moment when no external field is applied. Each magnetic domain has a well defined magnetic moment.

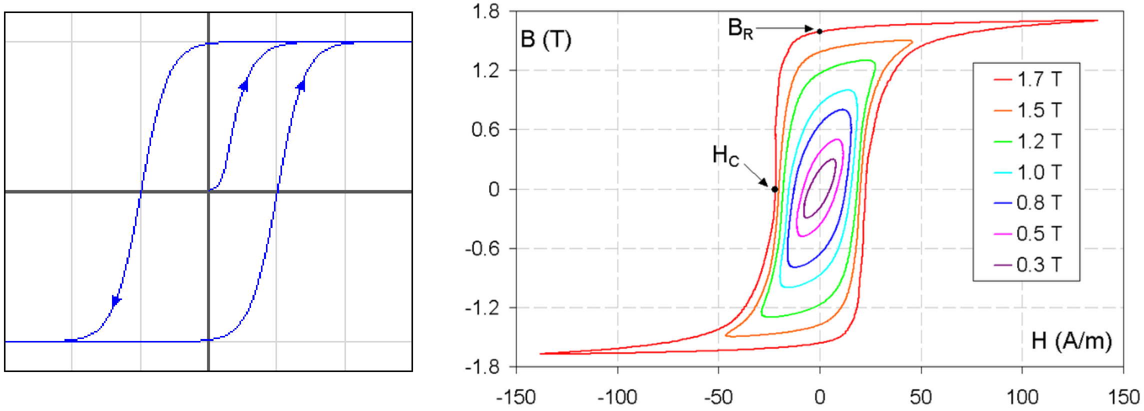


Several grains of NdFeB with magnetic domains

When a magnetic field, H, is applied, the magnetic domains will: (i) increase their volume if they have the same orientation as $\vec{H}$ and (ii) will rotate over the orientation of $\vec{H}$. This process of magnetization continues until all the magnetic moments will have the same orientation as $\vec{H}$. This state is named saturation and field for which this state is reached is named saturation field, see the figure bellow. The domains will remain re-oriented (state named “remanent”) when the field is turned off, thus creating a "permanent" magnet.



The magnetization as a function of the external field is described by a hysteresis curve:



A family of B-H loops

Hysteresis loops: magnetization (M) as function of magnetic field strength (H); (B_R denotes remanence and H_C is the coercivity.

Although this state of aligned domains for H=0 is not a minimal-energy configuration, it is extremely stable and has been observed to persist for millions of years in seafloor magnetite aligned by the Earth's magnetic field (whose poles can thereby be seen to flip at long intervals). The net magnetization can be destroyed by heating and then cooling (annealing) the material without an external field.

This property to keep magnetization basically indefinitely, until an external field is applied, is used to build non-volatile memories named MRAM.

Curie temperature

As the temperature increases, thermal motion, or entropy, competes with the ferromagnetic tendency for dipoles to align. When the temperature rises beyond a certain point, called the **Curie temperature appears a phase transition** and the system can no longer maintain a spontaneous magnetization, although it **still responds paramagnetically to an external field** – Curie Law.

Bellow there is a representative selection of ferromagnetic materials, ferrimagnetic (with *) and antiferromagnetic materials (underlined), along with their Curie (Néel) temperatures above which they cease to exhibit spontaneous magnetization.

Co – 1388 K, Fe – 1043 K, $FeOFe_2O_3^*$ - 858 K, $NiOFe_2O_3^*$ - 858 K, $CuOFe_2O_3^*$ - 728 K, $MgOFe_2O_3^*$ - 713 K, MnBi – 630 K, Ni – 627 K, MnSb – 587 K, $MnOFe_2O_3^*$ - 573 K, $Y_3Fe_5O_{12}^*$ - 560 K, CrO_2 – 386 K, MnAs – 318 K, Gd – 292 K, Dy – 88 K, EuO – 69 K

Making a magnet

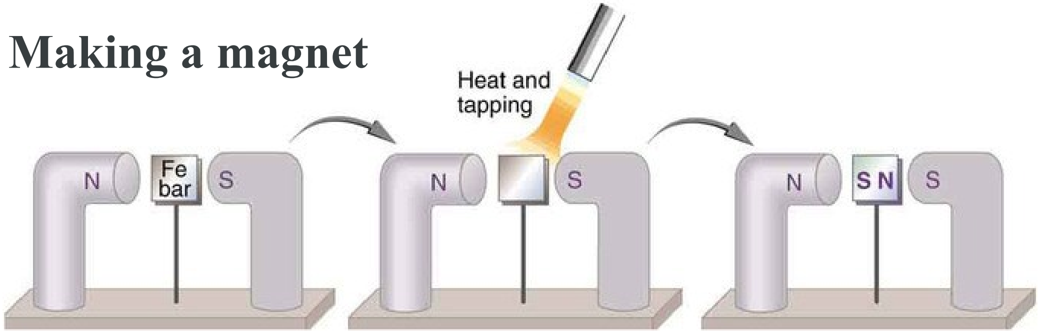
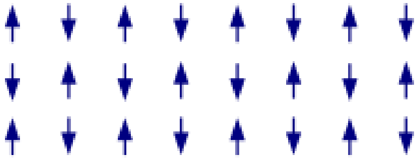


Image from https://learn.saylor.org/mod/page/view.php?id=35451

An unmagnetized piece of iron is placed between two magnets, heated above Curie temperature, and then cooled, or simply tapped when cold. The iron becomes a permanent magnet with the poles aligned as shown.

Antiferromagnetism

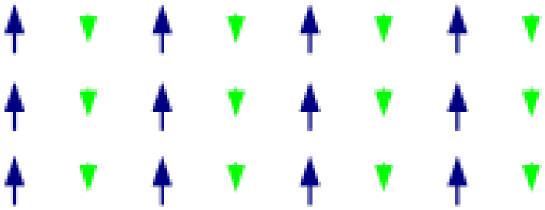


In materials that exhibit **antiferromagnetic** behaviour, the magnetic moments of atoms or molecules, usually related to the spins of electrons, align in a regular pattern with neighboring spins (on different sublattices) pointing in opposite directions. This is, like ferromagnetism and ferrimagnetism, a manifestation of ordered magnetism.
Generally, antiferromagnetic order may exist at sufficiently low temperatures, vanishing at and above a certain temperature, the Néel temperature (named after Louis Néel, who had first identified this type of magnetic ordering).
Above the Néel temperature, the material is typically paramagnetic.

Antiferromagnetic materials

Antiferromagnetic materials occur commonly among transition metal compounds, especially oxides. An example is the heavy-fermion superconductor URu_2Si_2. Better known examples include hematite, metals such as chromium, alloys such as iron manganese (FeMn), and oxides such as nickel oxide (NiO), Co oxide with T_{Neel}=291 K.

Ferrimagnetism



A **ferrimagnetic** material is one in which the magnetic moments of the atoms on different sublattices are opposed, as in antiferromagnetism; however, in ferrimagnetic materials, the opposing moments are unequal and a spontaneous magnetization remains. This happens when the sublattices consist of different materials or ions (such as Fe^{2+} and Fe^{3+}). Ferrimagnetism is exhibited by ferrites and magnetic garnets. The oldest-known magnetic substance, magnetite (iron(II,III) oxide; Fe_3O_4), is a ferrimagnet; it was originally classified as a ferromagnet before Néel's discovery of ferrimagnetism and antiferromagnetism in 1948.

Some ferrimagnetic materials are YIG (yttrium iron garnet) and ferrites composed of iron oxides and other elements such as aluminum, cobalt, nickel, manganese and zinc.

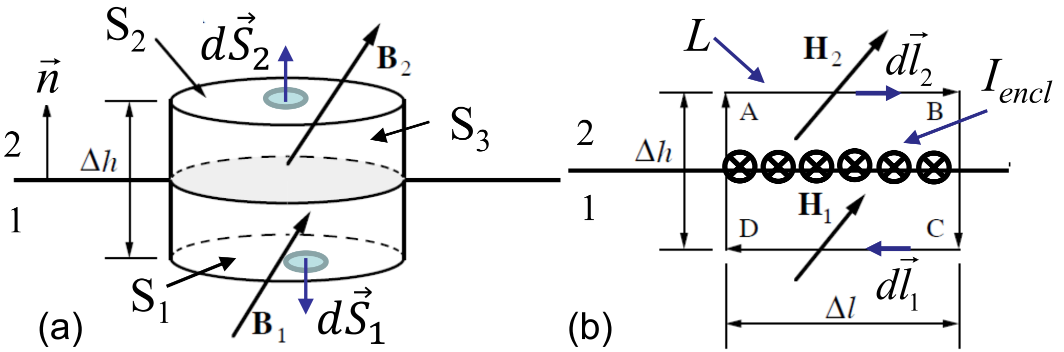
Magnetic materials can be characterised using different techniques like Hysteresisgraph, Vibrating Sample Magnetometer, Hall effect magnetometer, etc.

3.4.5. Solved applications

1. Find the boundary conditions for $\vec{H}$ and $\vec{B}$ at the interface (separation surface) between two different magnetic media. Also, find the refraction law for magnetic field lines at the separation surface between two different magnetic media.

Hint: Use Ampere's circuital law and Gauss' law for $\vec{H}$ and $\vec{B}$.

Consider the image below, adapted from https://newton.ex.ac.uk/teaching/CDHW/EM/CW970606-1.pdf



The following relations are true:

$$\nabla\vec{B}=0 \quad \text{and} \quad \oint_L \vec{H}d\vec{l}=I_{encl}$$

$$\mu_1=\mu_0\mu_{r1} \quad \text{and} \quad \mu_2=\mu_0\mu_{r2}$$

$$S_1=S_2=S \text{ and } dl_1=dl_2$$

S_3 – wrapping surface

$$\nabla\vec{B}=0 \Rightarrow \oiint_S \vec{B}d\vec{S}=0$$ - there are no magnetic charges

$$\oiint_S \vec{B}d\vec{S}=\iint_{S_1}\vec{B}_1 d\vec{S}_1+\iint_{S_2}\vec{B}_2 d\vec{S}_2+\iint_{S_3}\vec{B}d\vec{S}_3=0 \text{ with } \vec{B}_1=\vec{B}_{1n}+\vec{B}_{1t} \text{ and } \vec{B}_2=\vec{B}_{2n}+\vec{B}_{2t}$$

$$\vec{B}_1 d\vec{S}_1=\left(\vec{B}_{1n}+\vec{B}_{1t}\right)d\vec{S}_1=-B_{1n}dS_1 \text{ and } \vec{B}_2 d\vec{S}_2=\left(\vec{B}_{2n}+\vec{B}_{2t}\right)d\vec{S}_2=B_{2n}dS_2 \text{ with } dS_1=dS_2=dS$$

$$\Rightarrow\iint_{\Delta A}\left(-B_{1n}+B_{2n}\right)dS+\iint_{S_3}\vec{B}d\vec{S}_3=0;\text{when } \Delta h\to 0\Rightarrow\iint_{S_3}\vec{B}d\vec{S}_3\to 0\Rightarrow\boxed{B_{1n}=B_{2n};\mu_{r1}H_{n1}=\mu_{r2}H_{n2}}$$

- at the separation surface

To find the boundary conditions for the longitudinal components we shall consider circulation of the magnetic field strength along a closed contour L (b)

$$\oint_L \vec{H}d\vec{l} = I_{encl} \quad \text{with } I_{encl} = j_{lin}\Delta l\,;\, j_{lin} - \text{linear current density [A/m]}$$

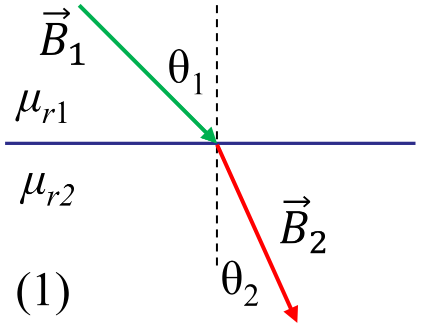
$$\vec{H}_1 = \vec{H}_{1n} + \vec{H}_{1t} \text{ and } \vec{H}_2 = \vec{H}_{2n} + \vec{H}_{2t}$$

Following the same approach, we find:

$$\int_{\Delta l}(H_{1t} - H_{2t})dl = \int_{\Delta l} j_{lin}dl \Rightarrow H_{1t} - H_{2t} = j_{lin} \Rightarrow H_{1t} - H_{2t} = 0 \text{ if } j_{lin} = 0$$

So, if there are no currents at the separation surface we have:

$$\boxed{H_{1t} = H_{2t} \Rightarrow \frac{B_{1t}}{\mu_{r1}} = \frac{B_{t2}}{\mu_{r2}}}$$



The refraction for magnetic field lines at the separation surface:

Assume that the incidence angle is θ_1 and the refraction angle is θ_2.

$$B_{1n} = B_{2n}\ ;\ B_{1n} = B_1\cos\theta_1\ ; B_{2n} = B_2\cos\theta_2 \Rightarrow \mu_{r1}H_1\cos\theta_1 = \mu_{r2}H_2\cos\theta_2 \quad (1)$$

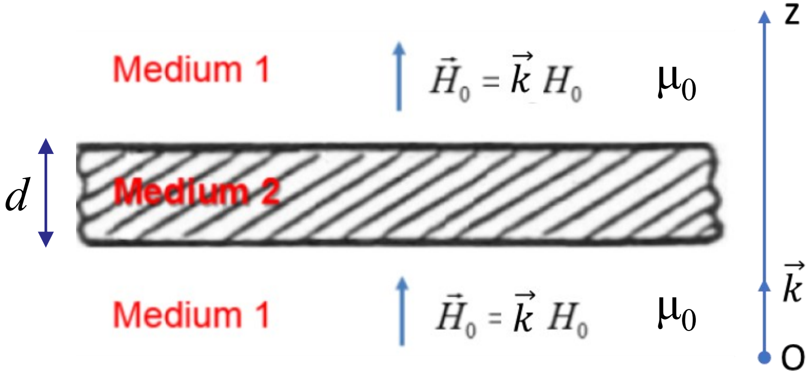
$$H_{1t} = H_{2t} \Rightarrow H_1\sin\theta_1 = H_2\sin\theta_2 \quad (2)$$

$$\frac{(2)}{(1)} \Rightarrow \boxed{\frac{tg\theta_1}{tg\theta_2} = \frac{\mu_{r1}}{\mu_{r2}}}$$

2. A very large slab of material of thickness d lies perpendicularly to a uniform magnetic field of intensity $\vec{H}_0 = \vec{k}H_0$. Ignoring edge effect, determine the magnetic field intensity in the slab:

a) if the slab has a permeability μ;

b if the slab is a permanent magnet having a magnetization vector $\vec{M}_i = \vec{k}M_i$.



a) We consider only the normal component of B

- from the boundary conditions:

$$B_{1n} = B_{2n} \Rightarrow \mu_0 H_0 = \mu H_2 \Rightarrow H_2 = \frac{H_0}{\mu_r}$$

with $\mu = \mu_0 \mu_r$

If μ_r=10 and H_0=100 kA/m ➔ H_2=10 kA/m

b) In the slab we have:

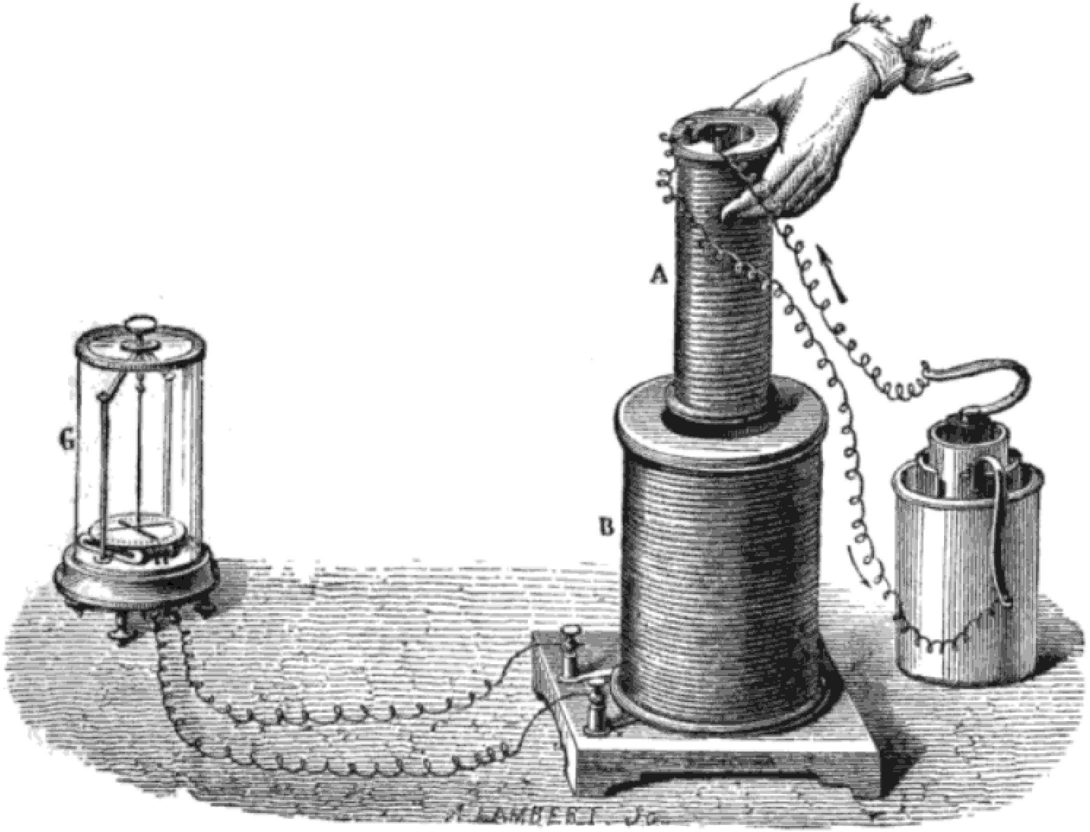
$$\vec{B} = \mu_0 \vec{H}_2 + \mu_0 \vec{M}_i \Rightarrow \mu_0 H_0 = \mu_0 H_2 + \mu_0 M_i$$

$$\Downarrow$$

$$H_2 = H_0 - M_i$$

If M_i=500 kA and H_0=100 kA/m ➔ H_2=-400 kA/m

Chapter IV. Electromagnetic induction



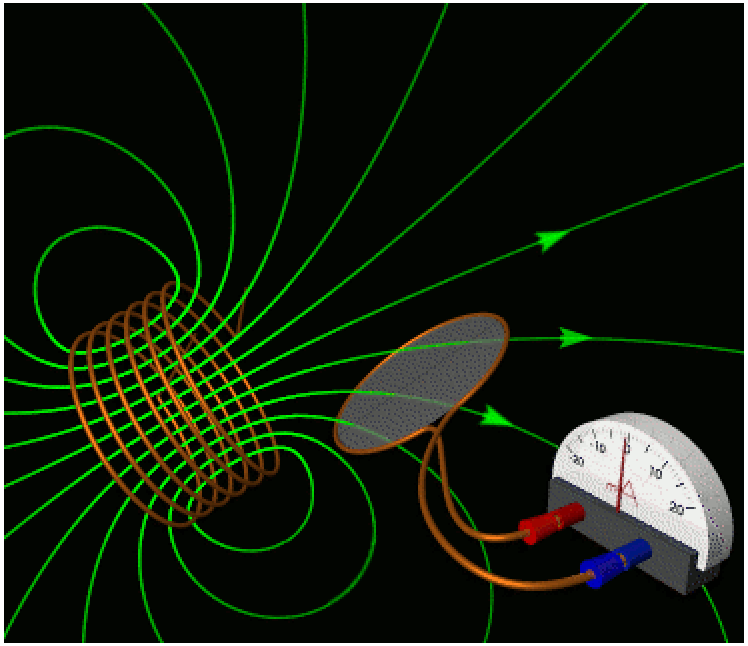
Faraday's experiment showing induction between coils of wire
https://books.google.com/books?id=JzBAAAAAYAAJ&pg=PA285

Extended Table of Contents

4.1. Introduction

In the previous chapters we discussed about static electric and magnetic fields and their effects on substance. Apparently these fields are not coupled. However, as we'll see starting from this chapter, in fact these fields are related in the case of time varying phenomena.

Electromagnetic induction is the production of an electromotive force (e.m.f.) across an electrical conductor in a changing magnetic field. Michael Faraday is generally credited with the experimental discovery of electromagnetic induction in 1831.
James Clerk Maxwell mathematically described it as Faraday's law of induction and later generalized to become one of the four Maxwell equations. Lenz's law describes the direction of the induced field.



https://en.wikipedia.org/wiki/Electromagnetic_induction

Some useful resources:

https://phet.colorado.edu/en/simulation/faradays-law
https://phet.colorado.edu/en/simulation/legacy/faraday

4.2. Electromagnetic induction law

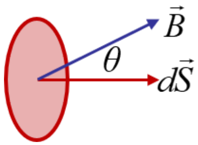
Electromagnetic induction was discovered by Michael Faraday, and published in 1831. It was discovered independently by Joseph Henry in 1832.

The electromotive force (e.m.f.) induced in a closed contour is:

$$U^{ind} = -\frac{d\Phi}{dt} \quad [\mathrm{V}]$$

$$\Phi = \iint_S \vec{B} d\vec{S} \quad [\mathrm{Wb}] \text{ - Weber}$$

From:



⇨ Φ – magnetic flux

In 1834 Heinrich Lenz formulated a law that gives the direction of the induced e.m.f. and current resulting from electromagnetic induction. Today it is known as the Lenz's Law:

> *The induced current is in such a direction as to <u>oppose</u> the magnetic flux variation causing it.*

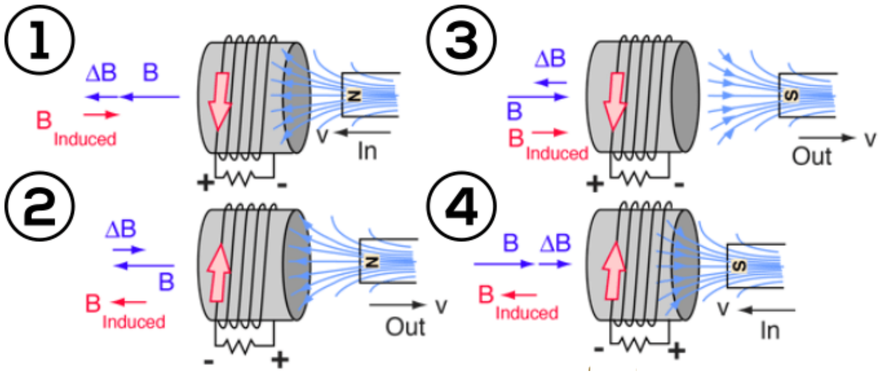


Illustration of Lenz's Law as stated above. As seen, the induced current changes the sign when the variation of the magnetic flux changes the sign.

Image from https://www.electrical4u.com/lenz-law-of-electromagnetic-induction/

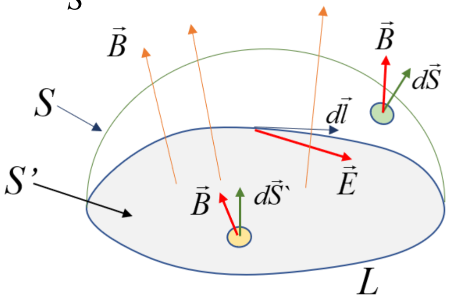
4.2.1. Differential form of the electromagnetic induction law

The Faraday’s law evidenced a new physical phenomenon for the 19th century: *a time varying magnetic field generates an electric field*.

More important, this shows that the electric field can be created not only by electric charges, but also by a time varying magnetic flux as well.

From: $U^{ind} = \oint_L \vec{E}d\vec{l}$ and $\Phi = \iint_S \vec{B}d\vec{S}$ $\Rightarrow$ $U^{ind} = \oint_L \vec{E}d\vec{l} = -\frac{d}{dt}\iint_S \vec{B}d\vec{S}$

where L is the contour that surrounds the surface S



Using Stokes theorem (see chapter 3.3.2) we find:

$$\oint_L \vec{E}d\vec{l} = \iint_S \left(\nabla\times\vec{E}\right)d\vec{S} = -\int_S \frac{\partial\vec{B}}{\partial t}d\vec{S} \Rightarrow \boxed{\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}} \quad (1)$$

- ***the differential form of Faraday’s law***

Comments:

From electrostatics we know that electric field originates or terminates on charges:

$$\nabla\vec{E} = \frac{\rho}{\varepsilon} \text{ or } \nabla\vec{D} = \rho \text{ and } \nabla\times\vec{\mathrm{E}}=0$$

With formula (1), Maxwell found that electric field can be produced, also, by a time varying magnetic field, and the electric field lines are rotational and perpendicular on $\vec{B}$.

4.3. The displacement current – Maxwell’s approach

1864 the English theoretical physicist Clerk Maxwell recognized the dilemma posed by the application of Ampere’s circuital law to a system of accelerated charges

We learned that *B* is unique determined by:

$$div\vec{B} = 0$$

$$rot\vec{B} = \nabla \times \vec{B} = \mu_0 \vec{j} \qquad (*)$$

- with *j* conduction current density

Also, the current density **must satisfy** the equation of continuity: $div\vec{j} + \frac{\partial \rho}{\partial t} = 0$

From (*) we have $div\vec{j} = \frac{1}{\mu_0} div\left(\nabla \times \vec{B}\right) \equiv 0$

But $\frac{\partial \rho}{\partial t} \neq 0$ for a system of moving charges

⇨ dilemma seen by Maxwell

The solution of this dilemma was posed by Maxwell: he redefines the current density by adding a new current named **the displacement current.**

From: $\rho = div\vec{D}$ ⇨ $div\vec{j} + \frac{\partial}{\partial t}\left(div\vec{D}\right) = 0$ ⇨ $div\left(\vec{j} + \frac{\partial \vec{D}}{\partial t}\right) = 0$

This formula agrees with: $div\vec{j}_{tot} = \frac{1}{\mu} div\left(\nabla \times \vec{B}\right) \equiv 0$ where $\vec{j}_{tot} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$

From this result we see that the total current density, j, has two components:

$$\boxed{\vec{j}_{tot} = \vec{j} + \frac{\partial \vec{D}}{\partial t}}$$ - the total current density

$$\vec{j} = \sigma \cdot \vec{E} \quad [\mathrm{A/m^2}]$$ - the *conduction current density*

$$\vec{j}_D = \frac{\partial \vec{D}}{\partial t} \quad [\mathrm{A/m^2}]$$ - the *displacement current density*

Now, we can write:

$$\nabla \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t} \qquad \nabla \times \vec{H} = \vec{j} + \varepsilon \frac{\partial \vec{E}}{\partial t}$$

or

$$\nabla \times \vec{B} = \mu \vec{j} + \mu\varepsilon \frac{\partial \vec{E}}{\partial t}$$

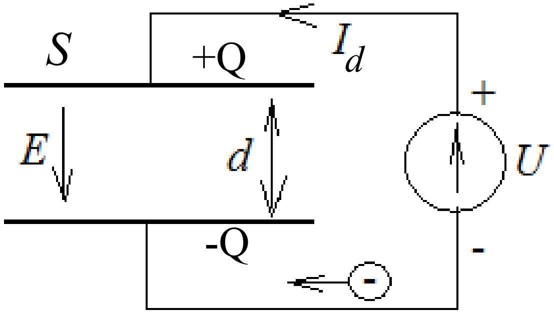
⇩

The displacement current creates a magnetic field just like conduction current does.

In vacuum: $\sigma = 0$ and $\mu = \mu_0$ ⇨ $\vec{j} = 0$ ⇨ $$\boxed{\nabla \times \vec{H} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}}$$

4.3.1. Physical description of the displacement current

Consider the process of charging or discharging of a capacitor ➔ a current, I_d, will flow when the charges (electrons) will be displaced through the source between the plates.



In 1864 the English theoretical physicist Clerk Maxwell noticed that a time varying electric field can produce a current, named *displaced current* which is different from the classical conduction current.

The following equations can be written as:

$$I_d = \frac{dQ}{dt}\ ,\ Q = C \cdot U \text{ and } U = E \cdot d$$

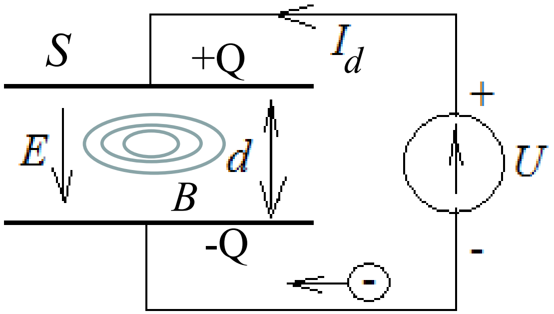
$I_d = j_d \cdot S$ where S is the area of the plate

$$I_d = C \cdot d\frac{dE}{dt},\quad C = \frac{\varepsilon S}{d} \Rightarrow I_d = d \cdot \frac{\varepsilon S}{d} \cdot \frac{dE}{dt} \Rightarrow I_d = S \cdot \varepsilon \frac{dE}{dt}$$

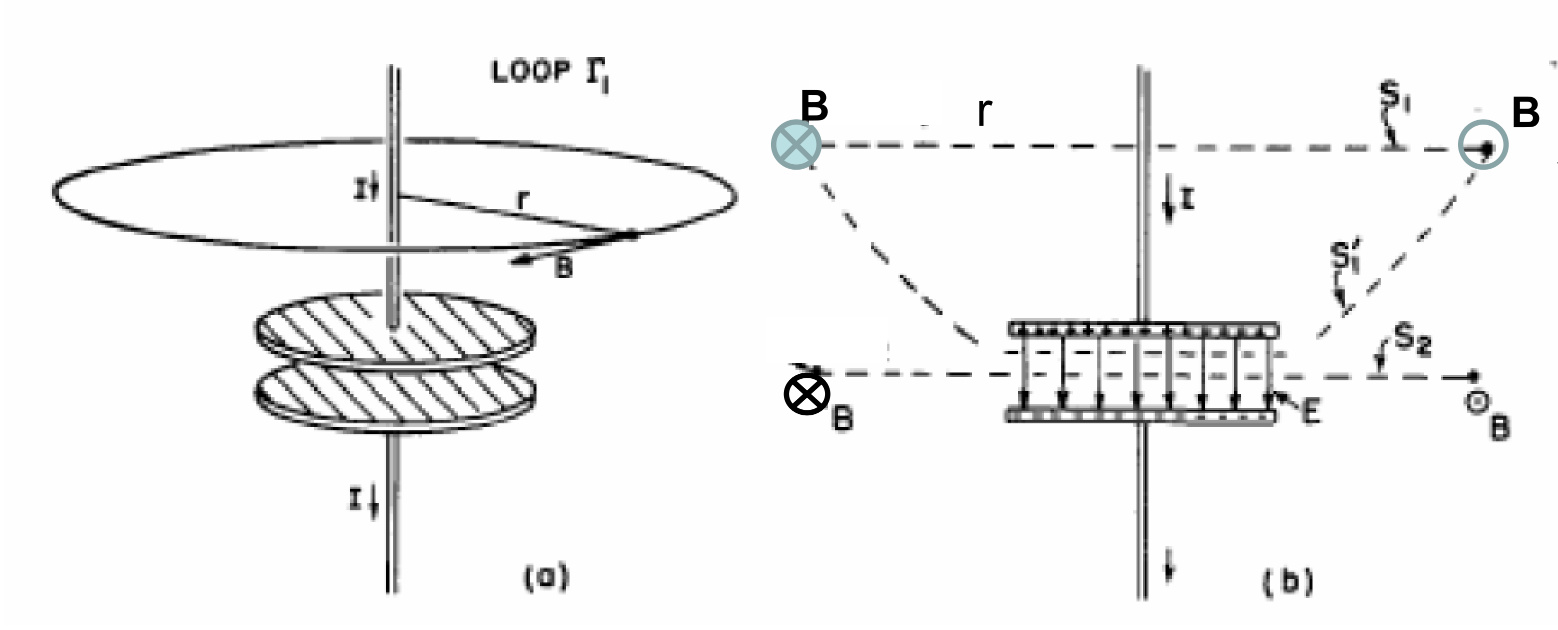
$$D = \varepsilon \cdot E \Rightarrow I_d = S \cdot \frac{dD}{dt} \Rightarrow \vec{j}_d = \frac{d\vec{D}}{dt}$$

The current I_d exists only if a displacement of charges exists, i.e. a charging or discharging process occurs: $d\vec{D}/dt \neq 0$. That means a time varying electric field should be present.

Generation of the magnetic field by the displacement current



In 1864 the English theoretical physicist Clerk Maxwell found, by theoretical calculations, that a time varying electric field can produce a magnetic field as it does, usually a conduction current.



$$B = \frac{\mu_0}{2\pi}\frac{I}{r}$$

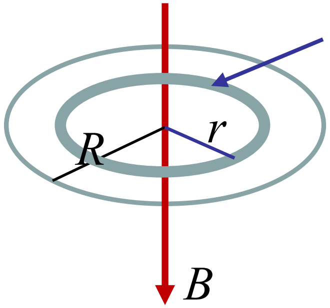
$$B = \frac{\mu_0}{2\pi}\frac{I_d}{r}$$

$$I = I_d$$

I_d produces a magnetic field like it does a conduction current ➔ **generation of e.m. waves.**

4.4. Solved and proposed applications

1. A circular loop with radius R consists of N tight turns of wire and is linked by an external magnetic field, $B = B_0\left(1-\frac{r}{2R}\right)\cos\omega t$ perpendicular to the plane of the loop; *r* is measured from the centre of the loop and ω is the pulsation of the magnetic field. Determine the induced EMF.



$$dS = 2\pi\cdot r\cdot dr$$

$$U = -\frac{d\Phi}{dt},\ \Phi = N\cdot\iint_S B\,dS \Rightarrow \Phi = N\cdot\int_0^R B(r)\cdot 2\pi\cdot r\cdot dr$$ - there are N surfaces!

$$\Phi = 2\pi B_0\cdot N\cdot\left[\int_0^R\left(1-\frac{r}{2R}\right)\cdot r\cdot dr\right]\cdot\cos\omega t = \frac{2}{3}\pi R^2 B_0\cdot N\cdot\cos\omega t$$

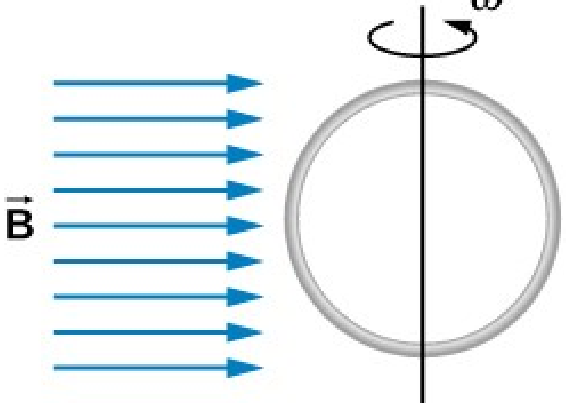
$$U = -\frac{d\Phi}{dt} = \frac{2}{3}N\pi R^2 B_0\omega\sin\omega t$$

2. A magnetic field B=β·t, where β=0.01 T/s and t is the time, is perpendicular to a circular metallic coil with 1 turn of radius R=10 cm placed in vacuum. Find the induced e.m.f. in the coil.

$$U = -\frac{d\Phi}{dt},\ \Phi = B\cdot\pi R^2 = \beta t\cdot\pi R^2 \Rightarrow U = -\beta\cdot\pi R^2 \Rightarrow U = -0.314\ \text{mV}$$

3. An alternating current $It)=I_0 cos\omega t$ is applied to a parallel-plate capacitor with circular plates of radius *R*, separated by a distance d. Determine *E(r,t)* and *B(r,t)* everywhere (neglecting the fringing field);

4. A circular loop of wire of radius R=10 cm is mounted on a vertical shaft and rotated at a frequency of n=5 cycles per second in a region of uniform magnetic field B=0.02 T perpendicular to the axis of rotation. (a) Find an expression for the time-dependent flux through the ring. (b) Determine the time-dependent current through the ring if it has a resistance of r=10 Ω.



a) $\Phi = \vec{B} \cdot \vec{S} = B \cdot S \cdot \cos\varphi \; , \; \varphi = \measuredangle\left(\vec{B}, \vec{S}\right)$

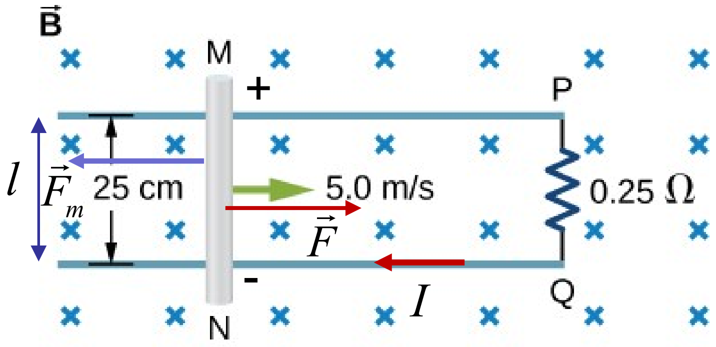
$$\text{at } t = 0 \Rightarrow \varphi_0 = \pi / 2 \Rightarrow \varphi = \omega t + \pi / 2$$

$$\omega = 2\pi n \Rightarrow \Phi = B \cdot \pi R^2 \cdot \cos\left(2\pi n \cdot t + \pi / 2\right)$$

b) $I = \frac{U}{r}, U = -\frac{d\Phi}{dt} = B \cdot \pi R^2 \cdot 2\pi n \cdot \sin\left(2\pi n \cdot t + \pi / 2\right)$

$$I = \frac{B \cdot \pi R^2 \cdot 2\pi n \cdot \sin\left(2\pi n \cdot t + \pi / 2\right)}{r} = \ldots$$

5. The conducting rod shown in the accompanying figure moves along parallel metal rails that are 25-cm apart. The system is in a uniform magnetic field of strength 0.75 T, which is directed into the page. The resistances of the rod and the rails are negligible, but the section PQ has a resistance of 0.25 Ω. (a) What is the emf (including its sense) induced in the rod when it is moving to the right with a speed of 5.0 m/s? (b) What force is required to keep the rod moving at this speed? (c) What is the rate at which work is done by this force? (d) What is the power dissipated in the resistor?



a) The sign is found using Lenz’s law

$$U = -\frac{d\Phi}{dt} = -\frac{BdS}{dt} = B \cdot l \cdot v \quad \text{because dS<0}$$

U=0.9375 V

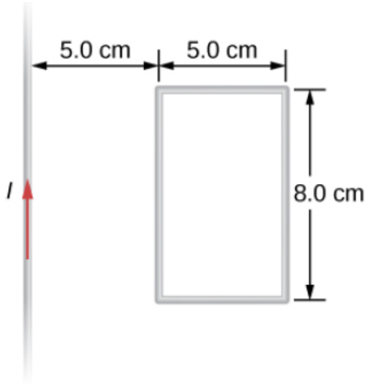
b) $\vec{F} + \vec{F}_m = 0 \Rightarrow F = F_m \Rightarrow F = B \cdot I \cdot l$

$$F = \frac{B^2 l^2 v}{R} = 0.703\ \mathrm{N}$$

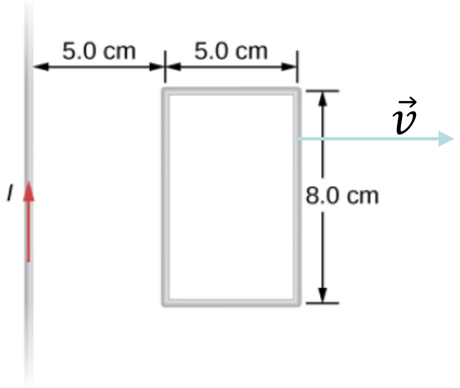
c) $P = F \cdot v = \dfrac{B^2 l^2 v^2}{R} = \ldots$

d) $P_R = \dfrac{U^2}{R} = \dfrac{B^2 l^2 v^2}{R} = \ldots$

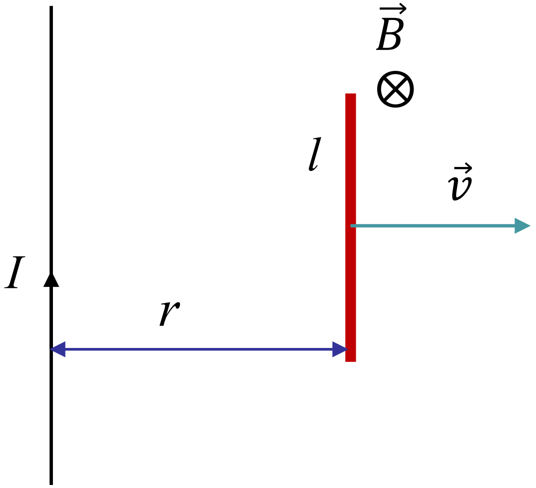
6. The current in long, straight wire (see the figure) is given by $I=I_0\sin\omega t$, where $I_0=15$ A and ω=120π rad/s. What is the amplitude of the current induced in the rectangular loop? The resistance of the loop is R=2 Ω



7. The current in the long straight wire is I=15 A. What is the current induced in the rectangular loop (l=8 cm, w=5 cm) which is moving with velocity v=2 m/s, at an instant t, in the illustrated position? The resistance of the loop is R=2 Ω. Can you draw an equivalent electrical circuit?

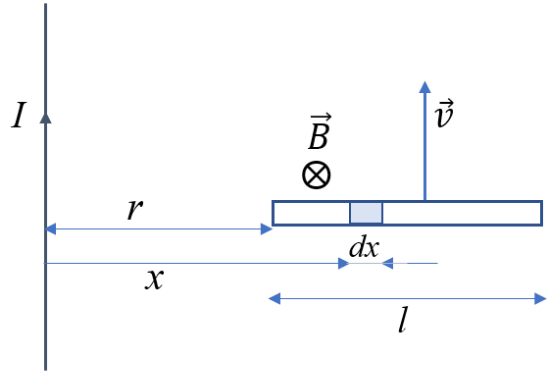


8. The current in the long straight wire is I=15 A. At t=0, the distance between the rod and wire is r_0=0.What is the induced e.m.f in the rectangular metallic rod, l=20 cm, which is moving with constant velocity v=10 cm/s after t=0.5 s?



$$U = Blv = \frac{\mu_0 I}{2\pi r} lv = \frac{\mu_0 I}{2\pi vt} lv = \frac{\mu_0 I}{2\pi t} l = 6 \cdot 10^{-6}\ \text{V}$$

9. The current in the long straight wire is I=15 A. What is the induced e.m.f in the rectangular metallic rod, l=10 cm, which is moving with velocity v=1 m/s ? The rod is at a distance r=1 cm from the wire. We can take ln11≈2.4

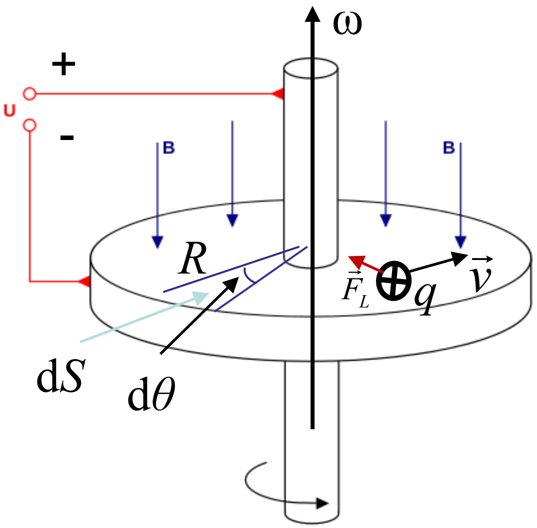


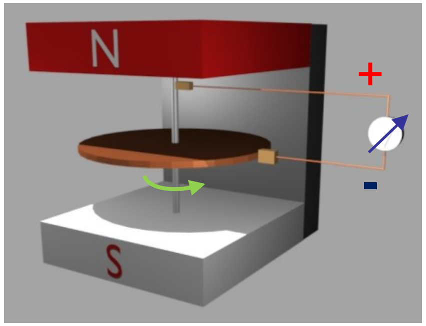
$$dU_x = B \cdot dx \cdot v = \frac{\mu_0 I}{2\pi x} \cdot v \cdot dx \Rightarrow U = \frac{\mu_0 I}{2\pi} \cdot v \int_r^{r+l} \frac{dx}{x}$$

$$U = \frac{\mu_0 I}{2\pi} \cdot v \cdot \ln \frac{r+l}{r} = 72 \cdot 10^{-6}\ \text{V}$$

4.5. Homopolar generator – Faraday disc

This is a special type of generator which delivers a constant DC voltage. His schematic construction is presented in the figures bellow.
Given B=1 T, R=10 cm and n=3000 rot/min, calculate |U|; find the polarity of the e.m.f source. Plot the voltage against the time



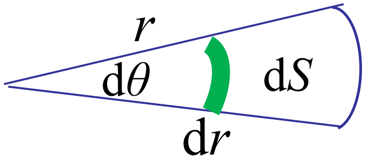


From the Lorentz force:

$$\vec{F}_L = q\vec{v}\times\vec{B}$$

⇩

polarity as shown in figure

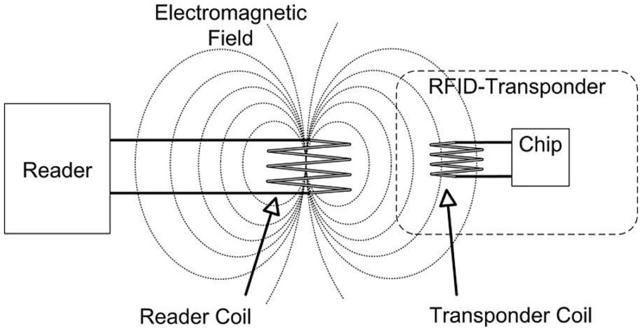


$$|U| = \frac{d\Phi}{dt} = \frac{BdS}{dt},\ dS = \left(\int_0^R r\,dr\right)d\theta = \frac{1}{2}r^2 d\theta$$

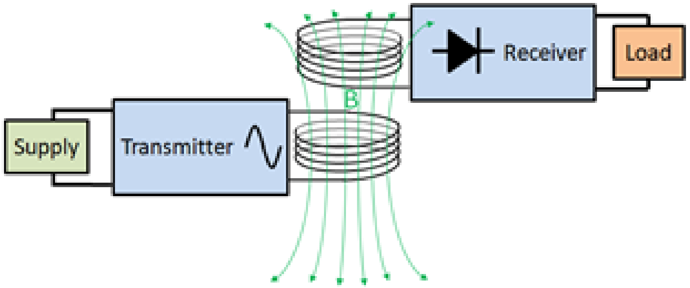
$$|U| = B\cdot\frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}Br^2\omega = 1.57\ \text{V}$$ which is constant in time

4.6. Typical applications of the e.m. induction phenomena

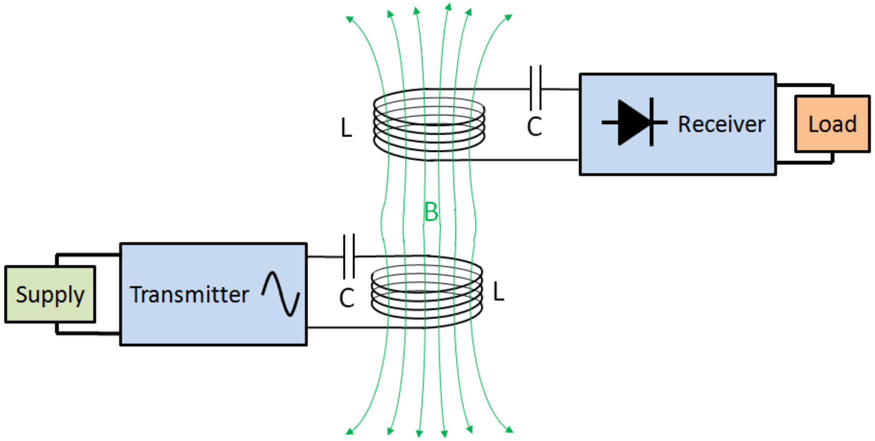
4.6.1. Energy transfer through e.m. induction phenomena



Reading RFID tags



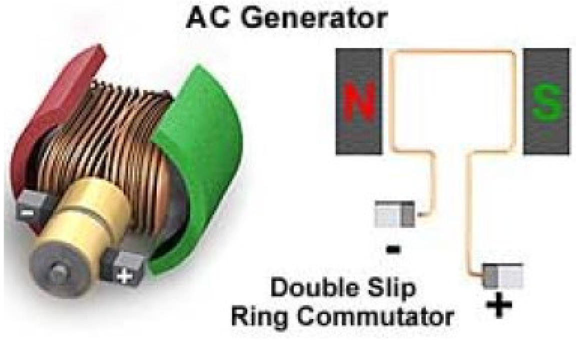
Electric transformer



Energy transfer using resonant circuits. They allow transmission of energy over larger distances.

4.6.2. Alternating Current (AC) power generator

The structure schematics of an A.C. power generator is presented bellow (see application 4, Section 4.4):



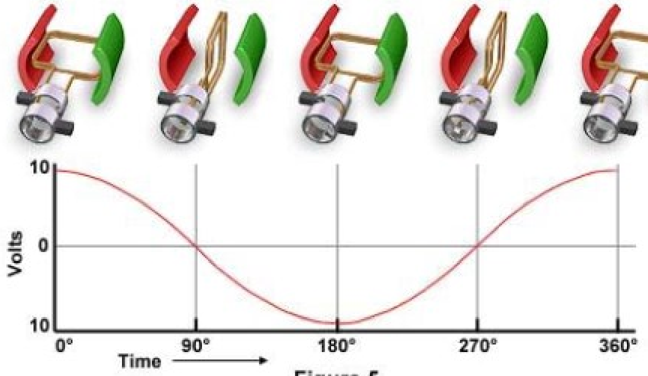
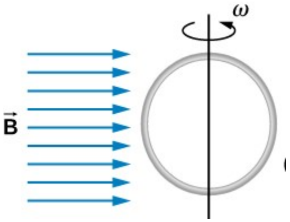


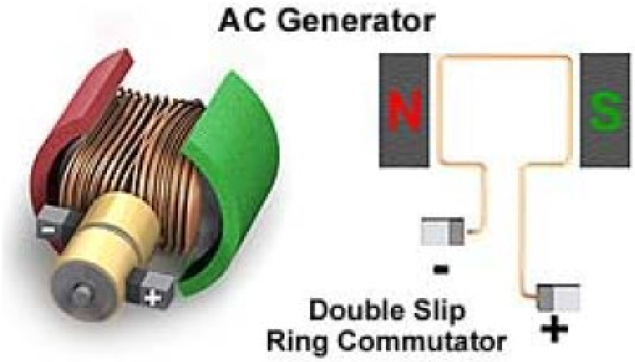
Figure 5

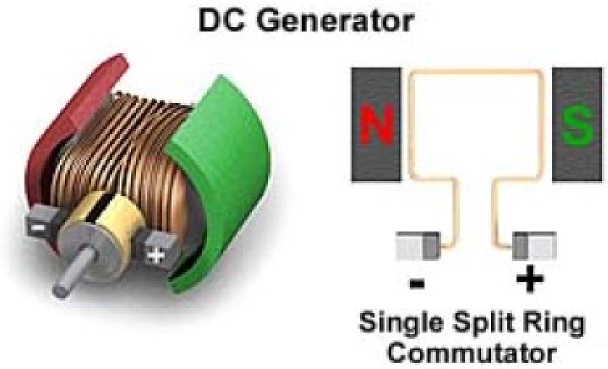


$$\Phi = NBS\cos\left(\omega t + \frac{\pi}{2}\right)$$

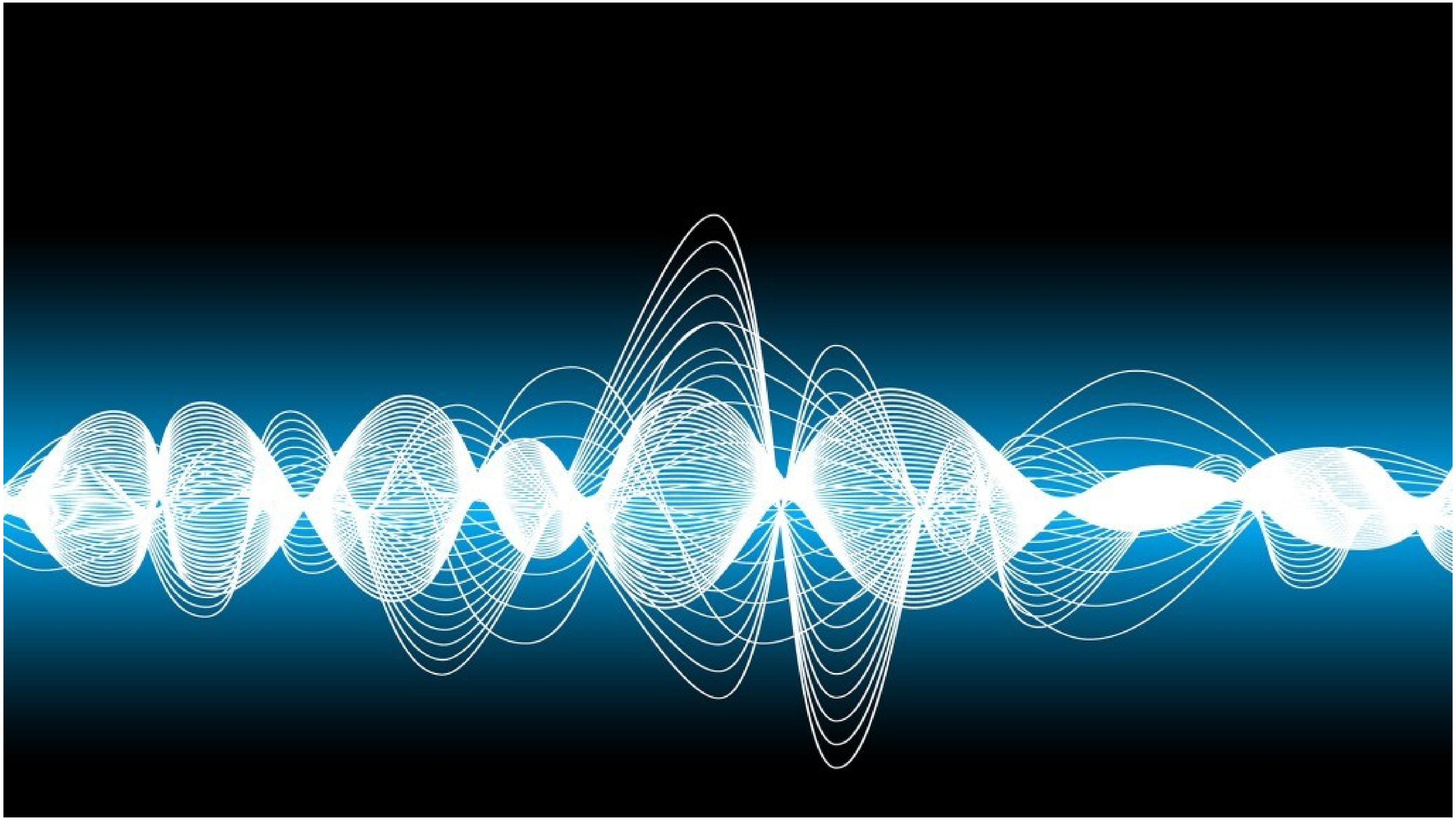
The time-dependent induced e.m.f. is: $e = -\frac{d\Phi}{dt} = NBS \cdot \omega \cdot \sin\left(\omega t + \frac{\pi}{2}\right)$

AC generator vs. Direct Current (DC) generator





Chapter V. Electromagnetic waves



https://nonstopengineering.blogspot.com/2016/11/EM-waves.html

Extended Table of Contents

5.1. Maxwell's Equations

In 1864 the English theoretical physicist James Clerk Maxwell found, by theoretical calculations the following equations that fully describe the electro-magnetic field - note that we gradually found these equations using a more descriptive method:

$$\nabla \times \vec{B} = \mu \vec{j} + \mu\varepsilon \frac{\partial \vec{E}}{\partial t}$$ - Maxwell-Ampère's circuital law

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$ - Faraday's law of the electromagnetic induction

$$\nabla \vec{E} = \frac{\rho}{\varepsilon} \qquad \nabla \vec{D} = \rho$$ - Gauss law for the electric flux

$$\nabla \vec{B} = 0$$ - Gauss law for the magnetic flux

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon \vec{E}$$
$$\vec{B} = \mu_0 \vec{H} + \vec{B}_i = \mu \vec{H}$$ - the material's equations

In dielectrics: $\varepsilon = ct. \quad \mu = ct.$ $\rho = 0 \Rightarrow \vec{j} = 0$

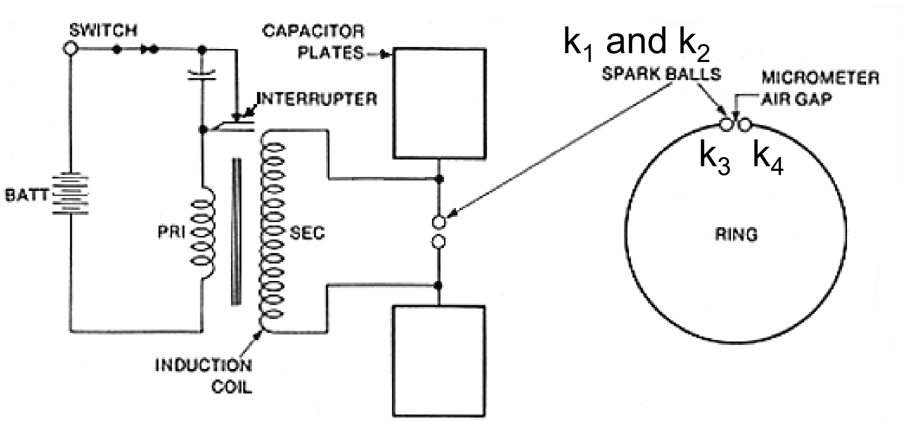
$$\nabla \times \vec{B} = \varepsilon\mu \frac{\partial \vec{E}}{\partial t} \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \vec{E} = 0 \qquad \nabla \vec{B} = 0$$

Maxwell’s equations predict that *electric and magnetic fields may exist in regions where no electric charges or currents are present*.

If the fields at one point of space vary with time, then some variation of the fields must occur at every other point of space at some other time. Thus changes in the electric and magnetic fields ***should propagate*** through space. The propagation of such a disturbance is called **electromagnetic wave** (the experimental proof was made in 1884 by Heinrich Hertz).

5.2. Early experiments on electromagnetic (e.m.) waves

- the existence and features of e.m. waves were theoretically described and predicted by James Clerk Maxwell, in 1864;
- the first experimental proof of this theory was given by Heinrich Hertz in 1888, ten years after Maxwell's death.

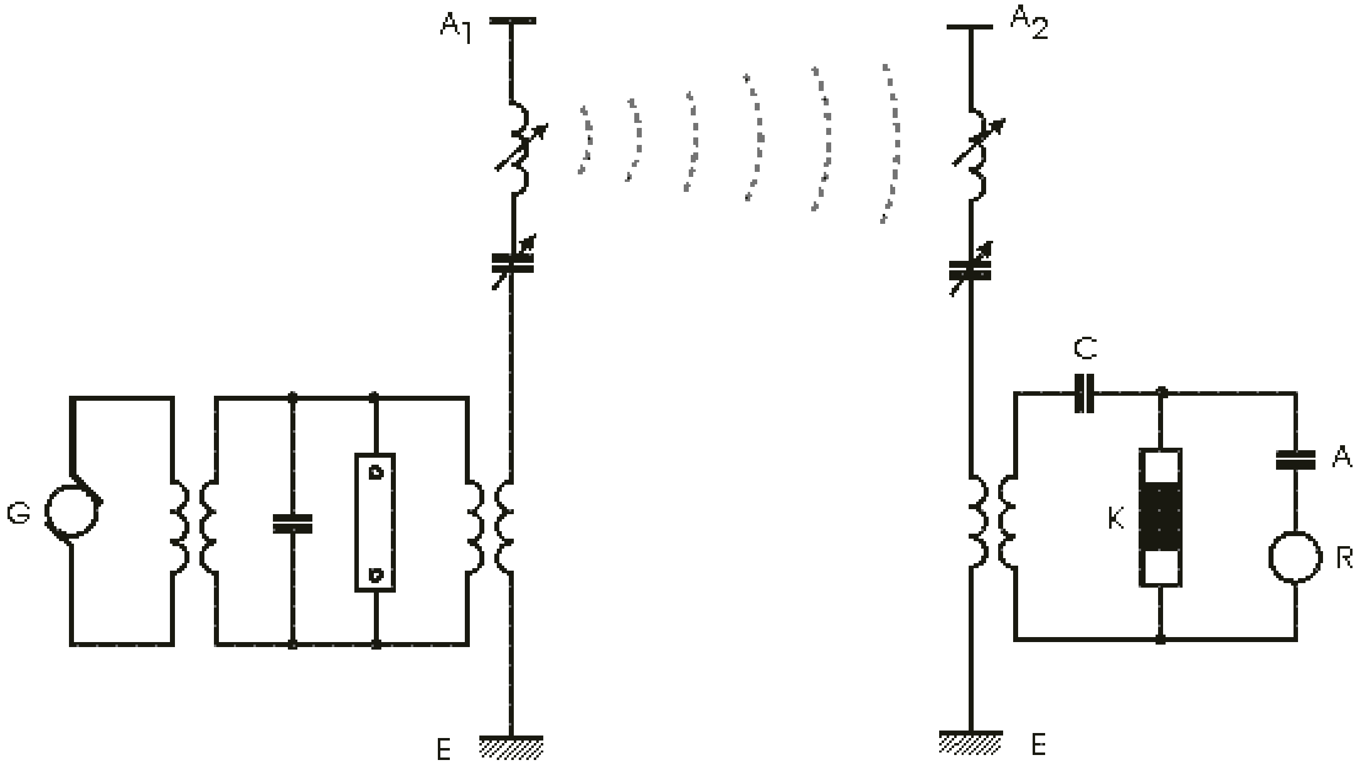


Detailed Schematic of Hertz's Experiment

- Hertz used an oscillatory circuit with a capacitor made of two plates;
- the "coil" was made of two straight conductors
- the plates could be moved along the conductors ➔ the capacitance of the circuit could be altered, and also its resonance frequency;
- with every interruption from the battery, a high voltage was produced at the output of the inductor, creating a spark between the narrow placed balls k_1 and k_2
- whenever there was a spark in the oscillator between the balls k_1 and k_2, a spark would also be produced by the resonator, between metallic balls k_3 and k_4.

Nikola Tesla demonstrated wireless broadcasting in 1893, at the Franklin Institute.

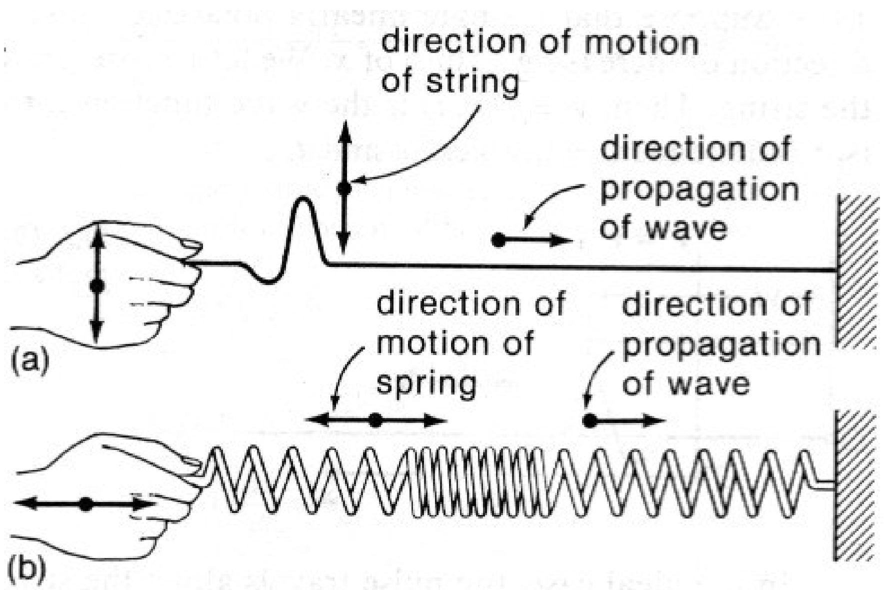
Tesla's idea was to produce electromagnetic waves by means of oscillatory circuits and transmit them over an antenna. A receiver would then receive the waves with another antenna and oscillatory circuit being in resonance with the oscillatory circuit of the transmitter. This represented the groundwork of today's radio communications.



5.3. Wave equations. General characteristics of waves

A disturbance that propagates in a given medium is named **wave**

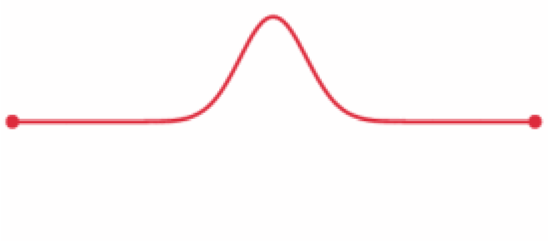
- one-dimensional
- two-dimensional
- three dimensional



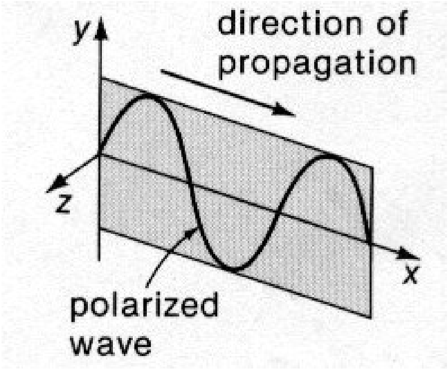
A transverse wave

A longitudinal wave

https://phet.colorado.edu/en/simulation/wave-on-a-string

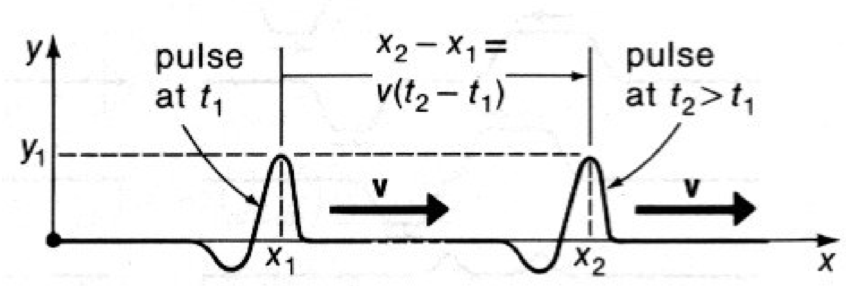


A **pulse** traveling through a string with fixed endpoints



A wave that is linearly polarized in the direction of the *y*-axis

5.3.1. The phase velocity and group velocity



$y = f(x - vt)$ describes the state in x and t

A wave pulse travels to the right with a velocity *v* along a taut string. The location of the pulse is shown at times t_1 and t_2.

To **give the same phase *u*₀** at these instants $u_0 = x_1 - vt_1 = x_2 - vt_2$

$$v = \frac{x_2 - x_1}{t_2 - t_1}$$

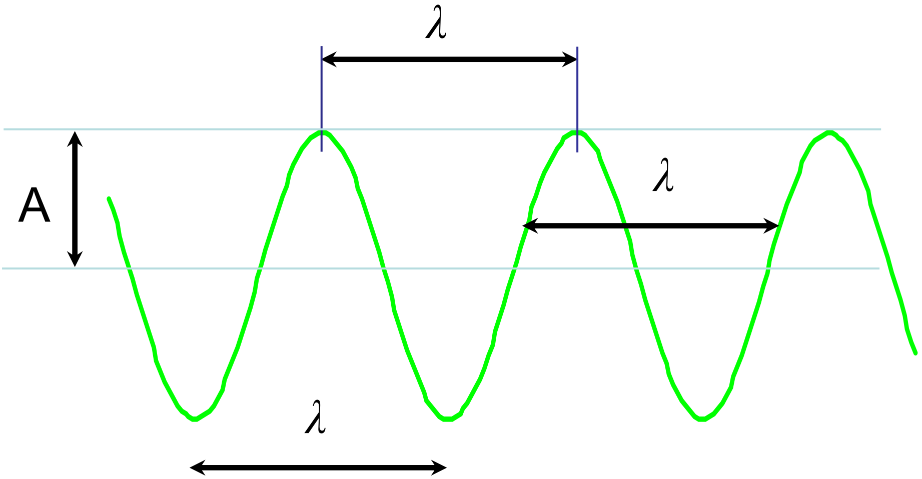
- phase u_0 of the pulse peak to be a constant, independent on time

$$\boxed{v = \frac{dx}{dt}}$$

- ***any feature** of the wave pulse has a coordinate location x that moves with* **a velocity** ***v** named **phase velocity***

There are two velocities that are associated with waves, the **phase velocity** and the **group velocity**.

Phase velocity



λ – wavelength, A - amplitude

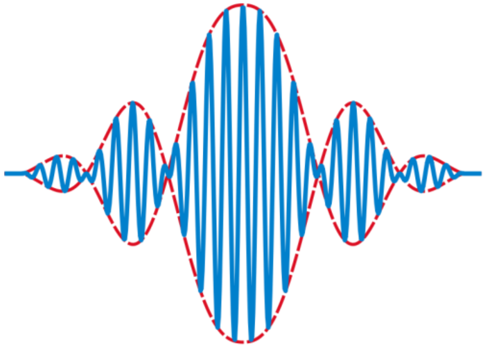
The wavelenght λ=v·*T* (T is time period, *T=1/ν*) is the shortest distance over which the wave repeats itself.

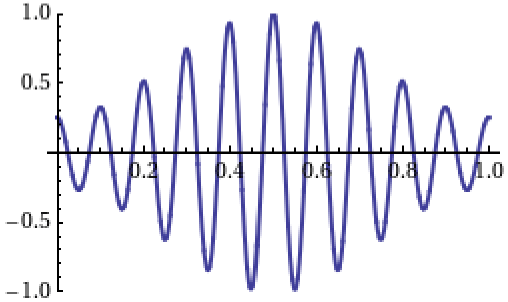
phase velocity

$$\boxed{v_{ph} = \frac{\omega}{k}, k = \frac{2\pi}{\lambda}}$$

k – wave number

Group velocity





The **group velocity** depends upon the dispersion relation connecting ω and k

$$\boxed{v_{gr} = \frac{d\omega}{dk}}$$

5.4. The general wave equation

The general equation that describes a wave is:

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$$

wave equation - waves that propagate in one dimension (*x*-direction)
v – wave velocity

or

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$$

- in a three-dimensional medium

$\Psi(x,t)$ represents a generalized displacement from equilibrium (e.g. the displacement of a string, a pressure variation, electric or magnetic field variation, etc.).

5.4.1. Plane waves

$$\boxed{\psi(x,t)=\psi_0\sin(\omega t-kx+\phi)}$$

- solution of wave equation; sinusoidal wave

The solution is periodic in x and t.

$$\omega=2\pi\nu$$

ν – *frequency*

ϕ – *initial phase*

k=2π/λ - the wave number

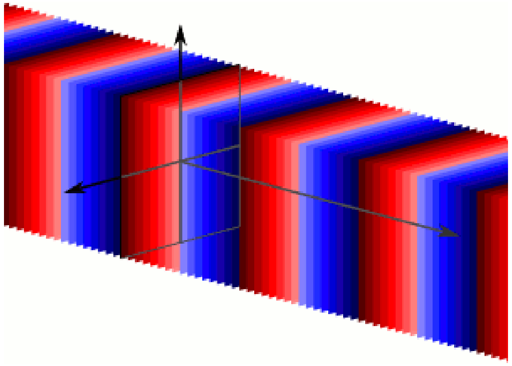
$\omega=k\cdot \mathrm{v}$ ω - pulsation

v – wave velocity

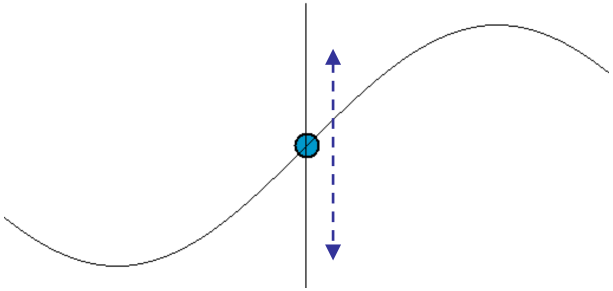
The wavelenght *λ*=v·*T* (T is time period, *T=1/ν*) is the shortest distance over which the wave repeats itself.

$$\boxed{\vec{\psi}(\vec{r},t)=\vec{\psi}_0\sin\left(\omega\cdot t-\vec{k}\cdot\vec{r}+\phi\right)\quad \vec{k}=\frac{2\pi}{\lambda}\vec{n}}$$

$\vec{n}$ - unit vector → direction of wave propagation

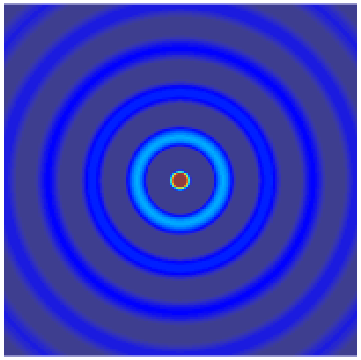


- a 3D plane wave; each color represents a different phase of the wave.

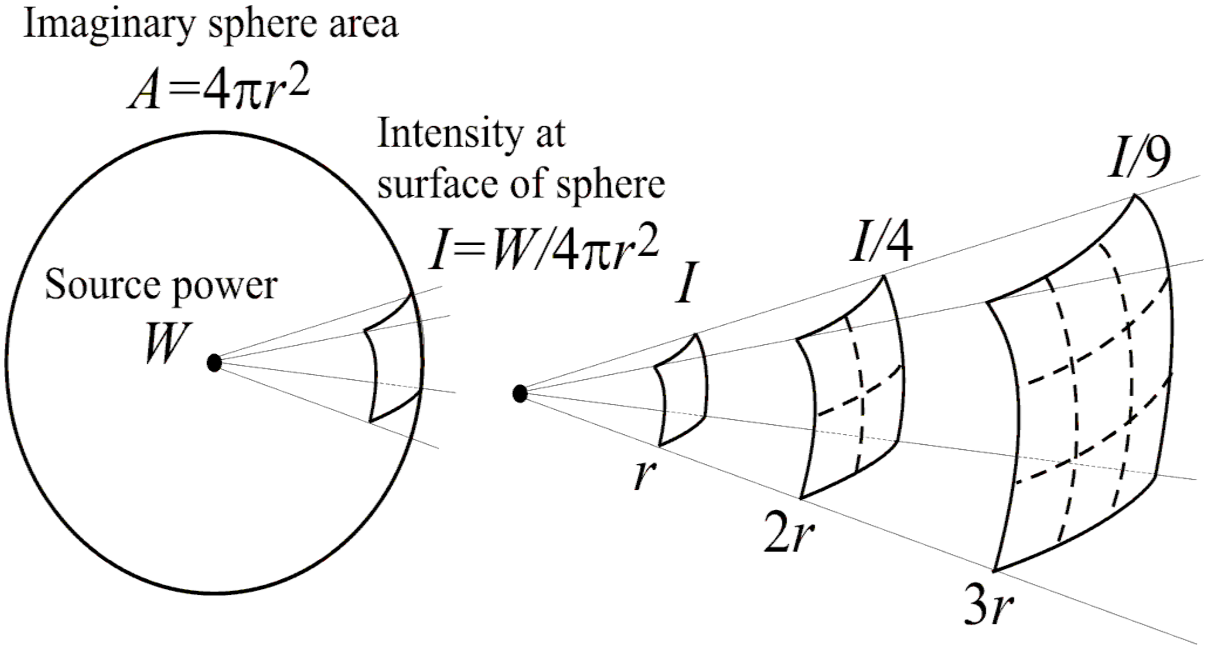


5.4.2. Spherical waves from a point source

$$\boxed{\vec{\psi}(\vec{r},t)=\frac{\vec{\psi}_0}{r}\sin\left(\omega\cdot t-\vec{k}\cdot\vec{r}\right)\quad \vec{k}=\frac{2\pi}{\lambda}\vec{n}}$$



$I\sim\frac{1}{r^2}$ - the wave intensity



https://phet.colorado.edu/en/simulation/waves-intro

5.5. Wave equations for electric and magnetic fields

For dielectric media we have:

$$\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times (\nabla \times \vec{H}) = \varepsilon \left(\nabla \times \frac{\partial \vec{E}}{\partial t} \right) = \varepsilon \frac{\partial}{\partial t} \left(\nabla \times \vec{E} \right)$$

$$\nabla \times (\nabla \times \vec{H}) = \nabla \left(\nabla \vec{H} \right) - \nabla^2 \vec{H} \qquad \nabla \vec{H} = 0$$

$$\vec{A} \times \left(\vec{B} \times \vec{C} \right) = \vec{B} \left(\vec{A} \cdot \vec{C} \right) - \vec{C} \left(\vec{A} \cdot \vec{B} \right)$$

But $\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$

$$\boxed{\Delta \vec{H} - \varepsilon\mu \frac{\partial^2 \vec{H}}{\partial t^2} = 0}$$

- ***these are wave equations*** **!**

In a similar way one can obtain:

$$\boxed{\Delta \vec{E} - \varepsilon\mu \frac{\partial^2 \vec{E}}{\partial t^2} = 0}$$

From:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}$$

$$\Downarrow$$

$$v^2 = \frac{1}{\varepsilon\mu}$$ - ***the speed of wave propagation in dielectric***

In vacuum: $c^2 = \frac{1}{\varepsilon_0 \mu_0} = 2.99792458 \cdot 10^8 \, m/s$ $\quad \varepsilon_0 = 8.85 \cdot 10^{-12}$ F/m and $\mu_0 = 4\pi \cdot 10^{-7}$ H/m

$$\frac{c}{v} = n = \sqrt{\varepsilon_r \mu_r}$$ - the index of refraction of the medium

5.5.1. Plane e.m. waves

We consider the following wave equations for the electric and magnetic fields components:

$$\vec{E} = \vec{E}_0 \sin(\omega t - kx)$$

$$\vec{H} = \vec{H}_0 \sin(\omega t - kx)$$

with $k=2\pi/\lambda$ - the wave number $\omega = 2\pi\nu$

$\omega = k\mathrm{v}$ - the pulsation; v is the wave velocity

Remember

The wavelenght $\lambda=\mathrm{v}\cdot T$ is the shortest distance over which the wave repeats itself (T is time period, $T=1/\nu$).

These solutions for wave equations are periodic both in x and t.

$\vec{E}_0 , \vec{H}_0$ - are the amplitudes of the electric and magnetic field components

In what follows we shall demonstrate that:

$$\boxed{\vec{H} \perp \vec{n}, \vec{E} \perp \vec{n}}$$

- *the electromagnetic waves are transverse waves*

$\vec{n}$ - unit vector → direction of wave propagation

and $$\boxed{\frac{E}{H} = \sqrt{\frac{\mu}{\varepsilon}}}$$

- the amplitudes of the electric and magnetic fields are related

To describe the waves, we can use complex notation; the imaginary part describes our waves:

$$\vec{E} = \vec{E}_0 e^{i(\omega t - kx)} \qquad e^{i\alpha} = \cos\alpha + i\sin\alpha,\ i = \sqrt{-1},\ k = \frac{2\pi}{\lambda}$$

$$\vec{H} = \vec{H}_0 e^{i(\omega t - kx)}$$ - this plane wave propagates along the Ox axis

We apply the ∇ operator, $\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$ to $\vec{E}$ and $\vec{H}$ waves

$$\nabla\vec{E} = \vec{i}\frac{\partial}{\partial x}\vec{E}_0 e^{i(\omega t - kx)} = -(ik)\vec{E}_0 e^{i(\omega t - kx)}\vec{i} = -(ik)\vec{i}\cdot\vec{E} \Rightarrow \boxed{\nabla = \vec{i}\frac{\partial}{\partial x} = -(ik)\vec{i}}$$ - <u>for *x* direction</u>

$$\boxed{\nabla = -(ik)\vec{n}}$$ - for an arbitrary direction of wave propagation, described by $\vec{n}$

Applying this operator to Maxwell's eqs.: $\nabla\vec{E} = 0$ in vacuum or dielectrics and $\nabla\vec{B} = 0$

$$\Downarrow$$

$$\nabla\vec{B} = -ik\mu\cdot\vec{n}\cdot\vec{H} = 0$$

$$\nabla\vec{D} = -ik\varepsilon\cdot\vec{n}\cdot\vec{E} = 0$$

$$\Rightarrow \boxed{\vec{H} \perp \vec{n},\ \vec{E} \perp \vec{n}}$$

$$\Downarrow$$

<u>*The electromagnetic waves are transverse waves*</u>

Now we show that: $\vec{E} \perp \vec{H}$

From the wave equation for $\vec{H}$ or $\vec{B} = \mu\vec{H}$ we find:

$$\frac{\partial \vec{B}}{\partial t} = \mu \frac{\partial \vec{H}}{\partial t} = \mu \vec{H}_0 i\omega e^{i(\omega t - kx)} = \mu\, i\omega \vec{H}$$

- in the same way: $\frac{\partial \vec{D}}{\partial t} = \varepsilon\, i\omega \vec{E}$

From Maxwell's equations in dielectrics: $\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t}$ $\quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

we have:

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \Rightarrow -ik\vec{n} \times \vec{H} = \varepsilon\, i\omega \vec{E}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow -ik\vec{n} \times \vec{E} = -\mu\, i\omega \vec{H}$$

$$\omega = k\mathrm{v}$$

$$\mathrm{v} = \frac{1}{\sqrt{\varepsilon\mu}}$$

⇨

$$-\left(\vec{n} \times \vec{H}\right) = \varepsilon\, \mathrm{v}\vec{E} = \frac{\varepsilon}{\sqrt{\varepsilon\mu}} \vec{E} = \sqrt{\frac{\varepsilon}{\mu}} \vec{E}$$

$$\left(\vec{n} \times \vec{E}\right) = \mu\, \mathrm{v}\vec{H} = \frac{\mu}{\sqrt{\varepsilon\mu}} \vec{H} = \sqrt{\frac{\mu}{\varepsilon}} \vec{H}$$

$$\boxed{\sqrt{\mu}\vec{H} = \sqrt{\varepsilon}\vec{n} \times \vec{E}}$$

⇨

$$\vec{E} \perp \vec{H} \perp \vec{n}$$

⇩

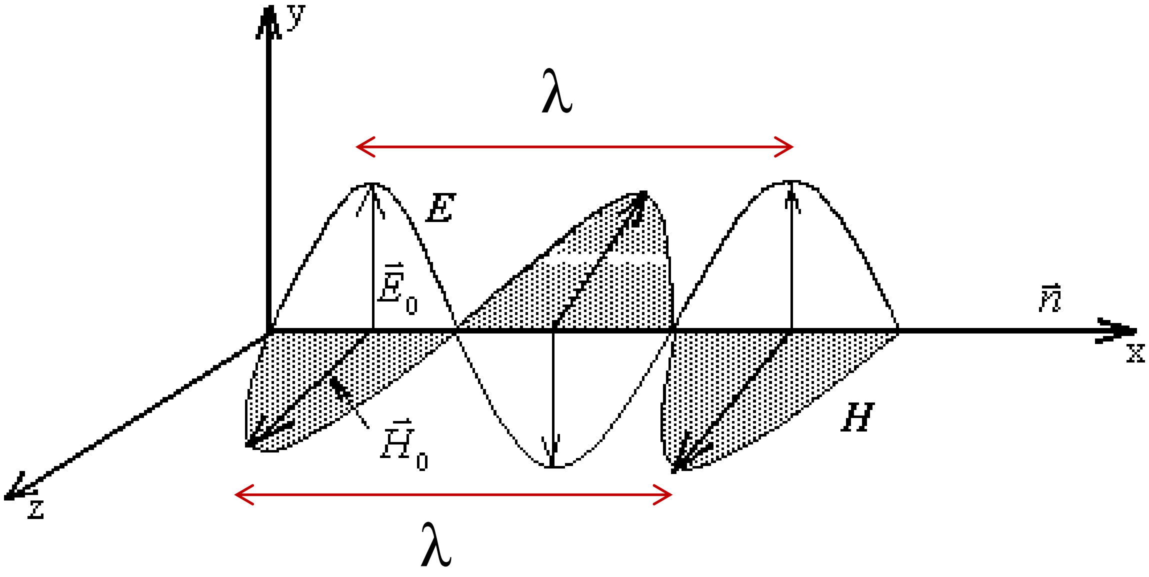
- the amplitudes of the fields are related: $\boxed{\frac{E}{H} = \sqrt{\frac{\mu}{\varepsilon}}}$

in vacuum: $\boxed{\frac{E}{H} = \sqrt{\frac{\mu_0}{\varepsilon_0}}}$

The results of the discussed equations are summarized and plotted bellow:

$$\vec{H} \perp \vec{n}, \vec{E} \perp \vec{n} \quad \text{and} \quad \vec{E} \perp \vec{H}$$

$\vec{E}$ and $\vec{H}$ components of the e.m. wave that propagates in a *dielectric media* *oscillate in phase* and their reciprocal orientations are shown bellow:



A plane polarized, monochromatic electromagnetic wave

- *plane polarized* - $\vec{E}$ and $\vec{H}$ components oscillate in a plane (xoy) and (xoz), respectively
- *monochromatic* – the wave has a single frequency of oscillation

5.6. Electromagnetic wave energy

The energy density of an electromagnetic wave can be expressed as:

$$\boxed{w = \frac{1}{2}(ED + HB) = \frac{1}{2}\left(\varepsilon E^2 + \mu H^2\right)}$$

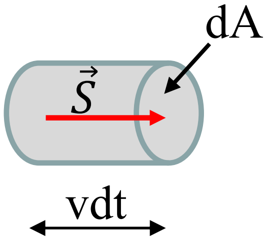
$$\boxed{\frac{E}{H} = \sqrt{\frac{\mu}{\varepsilon}}}$$

$$\Rightarrow \boxed{w = \varepsilon E^2 = \mu H^2} \quad [\mathrm{J/m^3}]$$

The intensity of an electromagnetic wave

$$S = \frac{1}{dA}\frac{dW}{dt} \quad [\mathrm{Jm^{-2}s^{-1}}] \qquad dW = w \cdot dA \cdot \mathrm{v} \cdot dt \Rightarrow \boxed{S = \mathrm{v} \cdot w}$$

$$\boxed{S = \frac{1}{\sqrt{\varepsilon\mu}}\mu H^2 = \sqrt{\frac{\mu}{\varepsilon}}H^2 = E \cdot H = \left|\vec{E} \times \vec{H}\right|} \quad [\mathrm{W/m^2}]$$



$$\boxed{\vec{S} = \vec{E} \times \vec{H}}$$

Poynting's vector

5.6.1. The average intensity of the wave

Using equations:
$$\begin{cases} \vec{E} = \vec{E}_0 \sin\left(\omega t - kx\right) \Rightarrow \vec{E} = \vec{E}_0 \sin\left(\frac{2\pi}{T}t - \frac{2\pi}{\lambda}x\right), \ \omega = \frac{2\pi}{T} \\ \vec{H} = \vec{H}_0 \sin(\omega t - kx) \Rightarrow \vec{H} = \vec{H}_0 \sin\left(\frac{2\pi}{T}t - \frac{2\pi}{\lambda}x\right), \ k = \frac{2\pi}{\lambda} \end{cases}$$

and $\left|\vec{S}\right| = \left|\vec{E} \times \vec{H}\right| = EH$

we can calculate *the time average intensity of the wave*:

$$\boxed{\overline{S} = E_0 H_0 \overline{\sin^2 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)} = \frac{1}{2} E_0 H_0 = E_{rms} H_{rms}}$$

where $\overline{\sin^2 2\pi \frac{t}{T}} = \frac{1}{T}\int_0^T \sin^2 2\pi \frac{t}{T} dt = \frac{1}{2}$ and $E_{rms} = \sqrt{\overline{E^2}} = \frac{1}{\sqrt{2}} E_0 \ , \ H_{rms} = \frac{1}{\sqrt{2}} H_0$

because both E and H behave like sine functions

*The **square root** of the **average square** of the electric field strength is called the root **mean** square (rms) field strength; the same is for magnetic field strength.*

The quantity $\int_A \left(\vec{E} \times \vec{H}\right) d\vec{A} = \frac{dW}{dt}$ is the flux of the Poynting's vector through a surface A

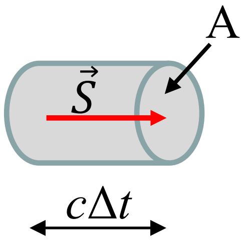
5.6.2. Electromagnetic momentum and radiation pressure

- *linear momentum density,* $\vec{G}$, *is defined* by: $\vec{G} = \frac{1}{c^2}\vec{S} = \frac{1}{c} w\cdot\vec{n}$, $[Js/m^4]$

We know that: $S = c\cdot w$ c is the speed of e.m. wave in vacuum

The total wave momentum contained within a volume dV=A·c·Δt will be absorbed by the surface:

$$G\cdot dV = \frac{1}{c} w\cdot A\cdot c\cdot\Delta t$$



So, a force F is exerted by the wave on an area of the surface A

$$F\Delta t = \frac{1}{c} w\cdot A\cdot c\cdot\Delta t$$

The force per unit of area is the *radiation pressure*, p_{rad}

$$p_{rad} = \frac{F}{A} = w \quad \left[N/m^2\right]$$

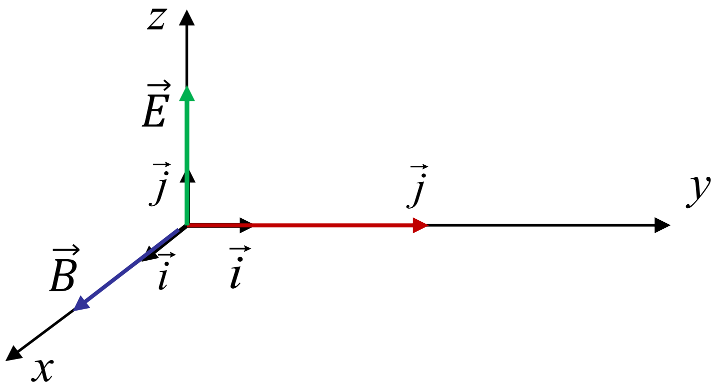
- **radiation pressure**

This formula has great importance for many applications; one of these represents the so called solar sailing concept for future space explorations.

5.7. Solved and proposed applications

1. A plane electromagnetic wave has the magnetic field component: $\vec{B}(t,y)=5\cdot10^{-6}\cos\left(10^{8}\pi\, t-\frac{\pi}{3}y\right)\vec{i}$

Find the frequency, wavelength and the speed of propagation of the wave. Also, find the electric field component.



- the wave propagates along y axis which means that $\vec{S}$ (the Poynting's vector) is along y axis;
- $\vec{B}$ oscillates along x axis
- by consequence, $\vec{E}$ oscillates along z axis
- rotating $\vec{E}$ over $\vec{B}$ we get direction of $\vec{S}$
- The following equations must be considered:

$$\frac{E}{H}=\sqrt{\frac{\mu}{\varepsilon}},\ \vec{S}=\vec{E}\times\vec{H}\ ,\ \vec{H}=\vec{H}_0\sin(\omega t-kx)\ \text{ and }\ \vec{B}=\mu\vec{H}$$

We find: $\omega=10^{8}\pi\ \text{rad/s},\ k=\frac{\pi}{3}\ \text{m}$

So: $\omega=2\pi f=10^{8}\pi\ \text{rad/s}\Rightarrow f=50\cdot10^{6}\ \text{Hz}$

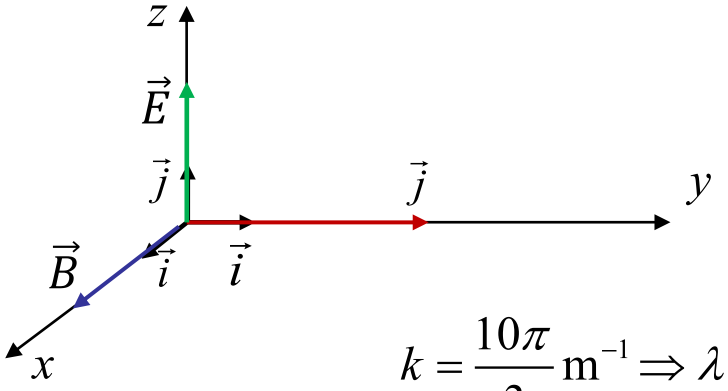
$$\lambda=\frac{2\pi}{k}\Rightarrow\lambda=6\ \text{m}\qquad\lambda=c\cdot T=\frac{c}{f}\Rightarrow c=f\cdot\lambda=3\times10^{8}\ \text{m/s}$$

$$E_0=H_0\sqrt{\frac{\mu}{\varepsilon}}=\frac{B_0}{\mu}\sqrt{\frac{\mu}{\varepsilon}}=\frac{B_0}{\sqrt{\varepsilon\mu}}=B_0\cdot c=1500\ \text{V/m}\Rightarrow \vec{E}=1500\cdot\cos\left(10^{8}\pi\, t-\frac{\pi}{3}y\right)\vec{k}\ \ \text{V/m}$$

2. A plane electromagnetic wave has the electric and magnetic fields coponents:

$$\vec{E}(t,y)=10^3\cos\left(10^9\pi\,t-\frac{10\pi}{3}y\right)\vec{k}\ \text{ V/m} \quad \text{and} \quad \vec{B}(t,y)=B_0\cos\left(10^9\pi\,t-\frac{10\pi}{3}y\right)\vec{i}\ \text{ T}$$

Find the frequency, wavelength and the speed of propagation of the wave. Also find the magnetic field amplitude, B_0, and the Poynting vector and intensity of the wave.



The solving steps are like for the Problem 1.
We find:

$$\omega=2\pi f=10^9\pi\ \text{rad/s}\Rightarrow f=500\cdot10^6\ \text{Hz}$$

$$k=\frac{10\pi}{3}\text{m}^{-1}\Rightarrow\lambda=\frac{2\pi}{k}=0.6\ \text{m},\quad \lambda=c\cdot T=\frac{c}{f}\Rightarrow c=f\cdot\lambda=3\times10^8\ \text{m/s}$$

$$\frac{E}{H}=\sqrt{\frac{\mu}{\varepsilon}}\Rightarrow B_0=\mu H_0=\mu\sqrt{\frac{\varepsilon}{\mu}}E_0=\frac{E_0}{c}=0.33\cdot10^{-5}\ \text{ T}$$

$$\vec{S}=\vec{E}\times\vec{H}=\frac{E_0B_0}{\mu}\cos^2\left(10^9\pi\,t-\frac{10\pi}{3}y\right)\vec{j}\ \text{ W/m}^2$$

$$\vec{S}(t,y)=\frac{0.33\cdot10^4}{2\pi}\cos^2\left(10^9\pi\,t-\frac{10\pi}{3}y\right)\vec{j}\ \text{ W/m}^2,\quad \bar{S}=\frac{1}{2}\frac{0.33\cdot10^4}{2\pi}\ \text{ W/m}^2$$

3. An AC electric field with frequency f=50 Hz generates a current through a conductor of resistivity $\rho=1.78\cdot10^{-8}\ \Omega\cdot\mathrm{m}$. Calculate the ratio between the displacement current and the conduction current. We assume that $\varepsilon=\varepsilon_0=8.854\times10^{-12}$ C/V m and the current is uniform distributed inside the conductor. What is this ratio if f=10 kHz?

4. Calculate the ratio between the energies transported by the electric and magnetic field components in an e.m. wave

5. A plane e.m. wave propagates in an isotropic media with $\varepsilon_r=3$ and $\mu_r=1$. The amplitude of the electric field component is $E_0=10$ V/m. Calculate:
a) the amplitude of the magnetic field component and b) the phase velocity of the e.m. wave.
(See problems 1 and 2)

6. The mean value of the intensity of the e.m. radiation which is coming from the Sun is 1353 W/m^2 (known as Solar constant). Calculate the mean value of the electric field component.

5.8. A classification of the e.m. waves

- $\lambda=10^{4}-10^{0}$ m – radio frequencies waves
- $\lambda=10^{0}-10^{-3}$ m – microwaves
- $\lambda=10^{-3}-8\cdot10^{-7}$ m – infrared radiation
- $\lambda=8\cdot10^{-7}-4\cdot10^{-7}$ m – visible radiation (light)
- $\lambda=4\cdot10^{-7}-10^{-8}$ m – ultraviolet radiation
- $\lambda=10^{-8}-10^{-12}$ m – X, γ radiation.

For visible region:

COLOUR	Wavelengths Range (nm)
Violet	400-450
Blue	450-500
Green	500-550
Yellow	550-600
Orange	600-650
Red	650-700

- there are no precisely defined boundaries between the bands of the electromagnetic spectrum; rather they fade into each other like the bands in a rainbow

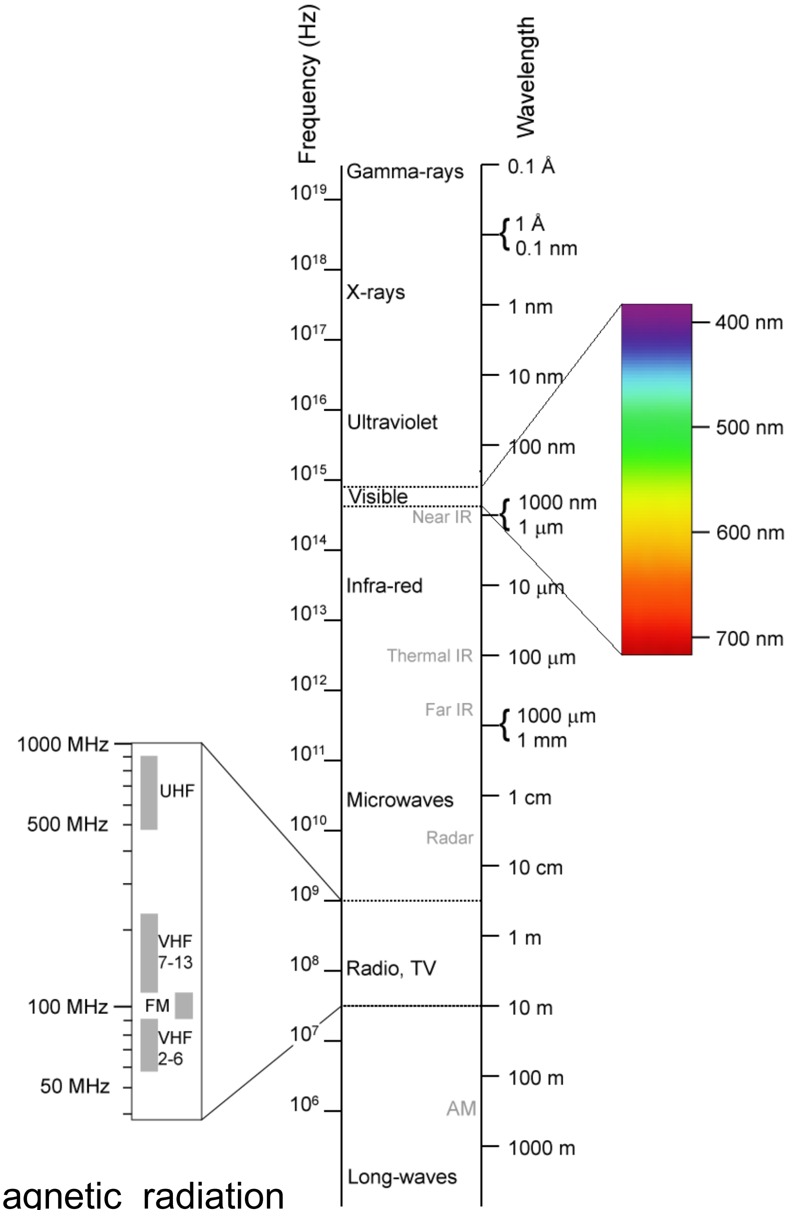
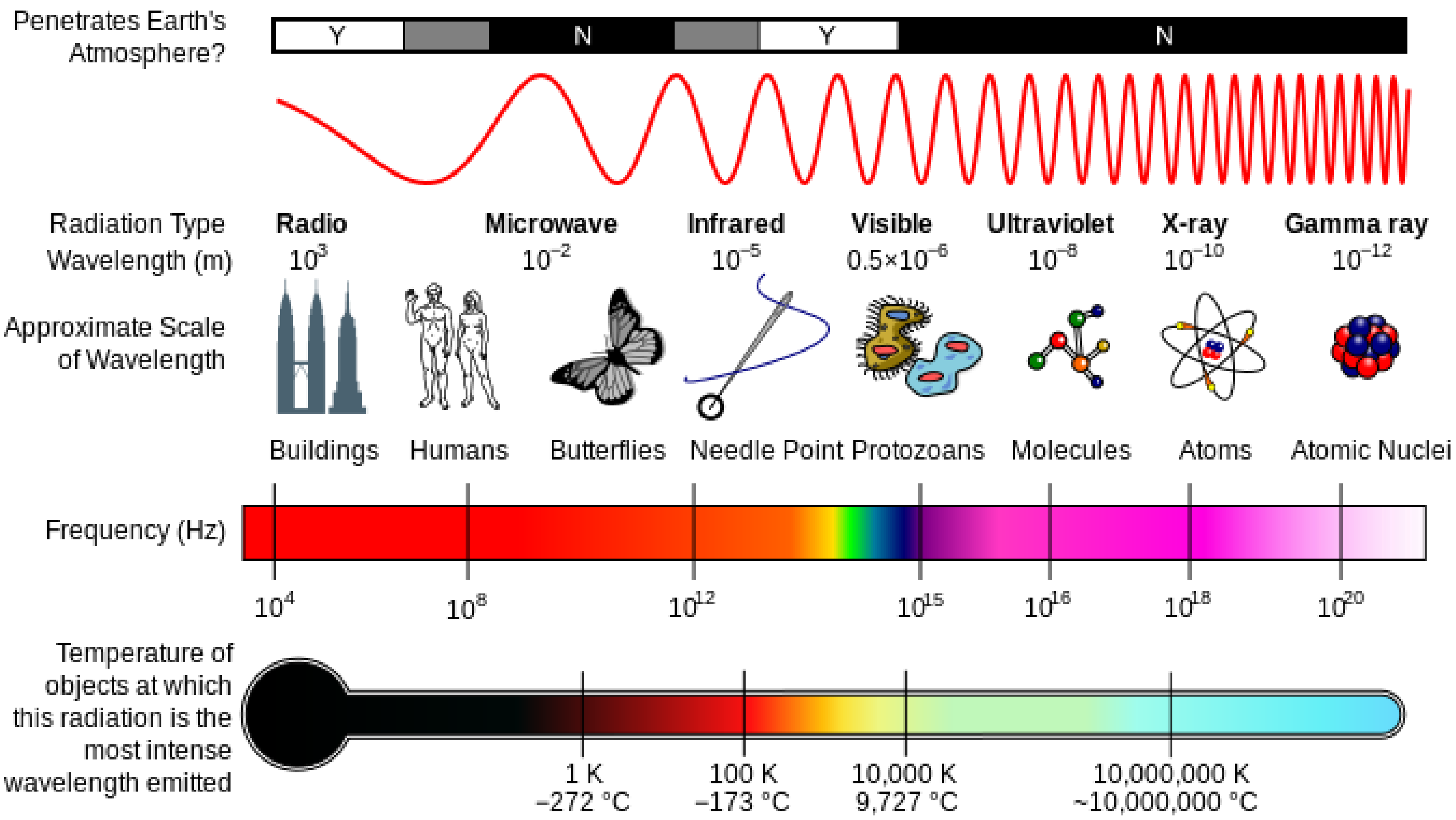


Image from: https://simple.wikipedia.org/wiki/Electromagnetic_radiation

Electromagnetic Radiation Spectrum (cont.)



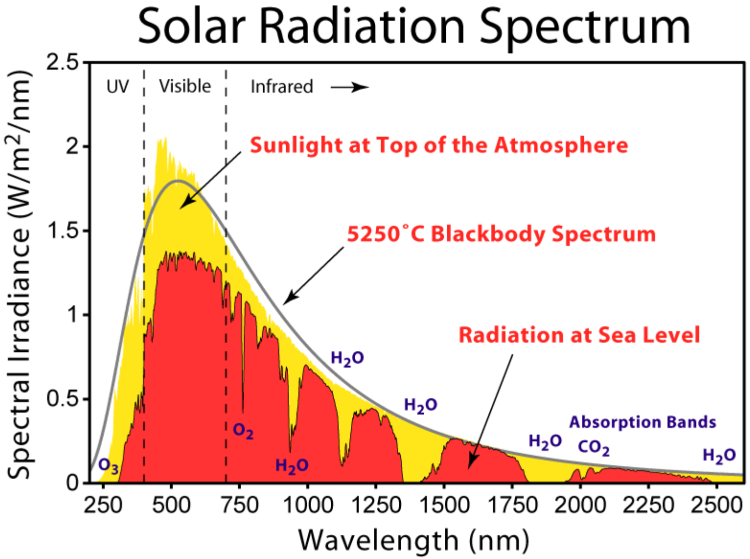
Comparing dimensions of objects with the wavelength of e.m. radiation and equivalent temperatures of objects that can emit radiation with a certain wavelength/frequency

https://en.wikiversity.org/wiki/Electromagnetic_radiation

5.9. Elements of photometry

The plot presents the solar radiation spectrum at top of the atmosphere and at the see level.
At the see level the spectrum is affected by absorption in gases molecules like H_2O, CO_2, etc.

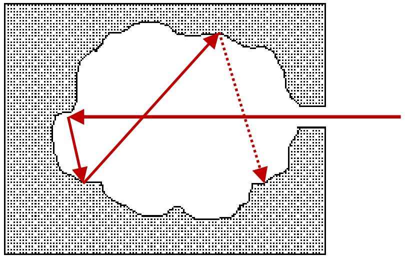
The sun produces light with a distribution similar to what would be expected from a 5525 K (5250 °C) object named “blackbody”, which is approximately the sun's surface temperature.



https://commons.wikimedia.org/wiki/File:Solar_Spectrum.png

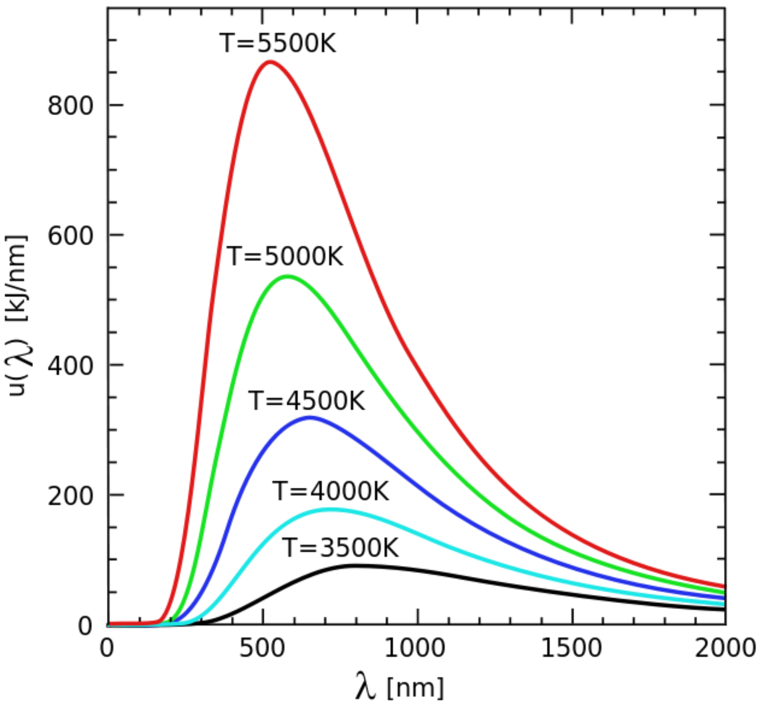
5.9.1. Blackbody radiation

For a ***shiny metallic surface***, the light isn't absorbed, it gets reflected.
For a ***black material*** like soot, *light and heat are almost completely absorbed*, and *the material gets warm*.
It is shown that *good absorbers of radiation are also good emitters*.



A cavity approximates a blackbody: the radiation that enters cannot escape, i.e. due to random reflection inside the cavity is entirelly absorbed by the irregular surface ➔ the temperature of the cavity increases

Observing the Black Body Spectrum



Black body thermal emission intensity as a function of wavelength for various absolute temperatures.

It is shown that following laws can be used to describe the black body radiation:

$$P = \sigma \cdot T^4, \ \sigma = 5.67\text{x}10^{-8} \, watts / m^2 \cdot K^4$$

Stefan Boltzmann's Law of Radiation

The area below the plot, for a given temperature, represents the total power of the emitted the radiation; this power is ~T^4.

$$\lambda_{\max} = \frac{3 \cdot 10^6}{T} \, [nm]; \ \nu_{\max} = \frac{a}{h} kT \approx 5.879 \cdot 10^{10} \cdot T \, [Hz]$$

Wien's Displacement Law

The wavelength, $\lambda_{\max}$, for which radiation intensity is maximum is ~1/T; by consequence, the frequency, $\nu_{\max}$~T.

These laws that describe the black-body radiation are used for temperature measurement when a direct contact between a thermometer and the system under investigation is not possible.

5.9.1. Photometric quantities

In what follows we briefly describe the main photometric quantities.

The radiant power

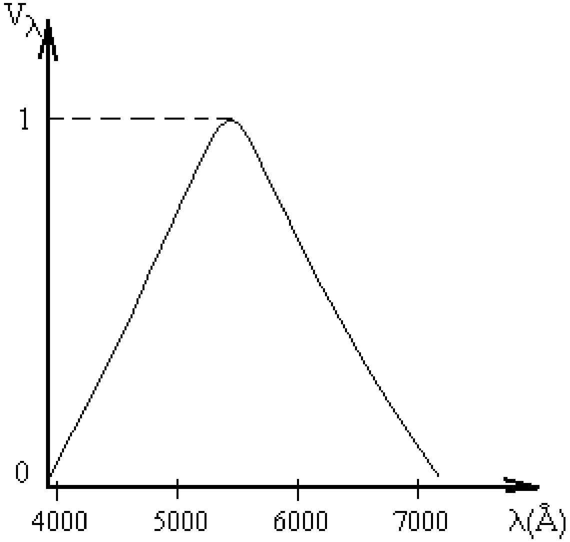
$$\boxed{P_\lambda = \int_A \left(\vec{E} \times \vec{H}\right) d\vec{A} = \frac{dW}{dt}}$$ [W]   with $\vec{S} = \vec{E} \times \vec{H}$, the wave intensity

Luminous power

$$\boxed{\Phi_\lambda = K V_\lambda P_\lambda}$$ [lm]   "Lumen"

K=683 lm/W and is called photometric factor

V_λ- spectral sensitivity of normal human eyes
V_λ=1 for λ=555 nm



A typical dependence of the human eyes sensitivity

A typical 100 watt incandescent bulb has a luminous power of about 1700 lumens.

Pointance or Intensity of Light

$$\boxed{I = \frac{\Phi_\lambda}{\Omega}}$$ [cd] “Candela”

Ω is the solid angle

Illumination

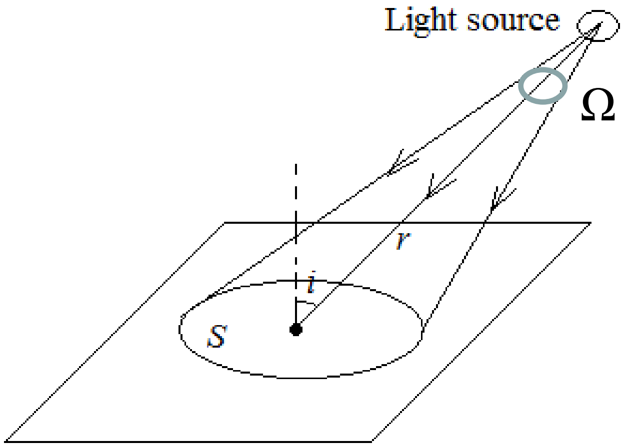
$$\boxed{E = \frac{\Phi_{inc}}{S}}$$ [lx] “Lux” Φ_{inc} is the flux of light striking the surface S

$$\Phi_{inc} = I \cdot \Omega \qquad \Omega = \frac{S_n}{r^2} = \frac{S \cdot \cos i}{r^2}$$

S_n is the surface normal to the light direction

$$\boxed{E = \frac{I \cdot \cos i}{r^2}}$$

- r is the distance from the source of light
- i is the incident angle



Luminous efficiency

$$\boxed{\eta = \frac{\Phi}{P}} \quad [\text{lm/W}]$$

The incandescent bulbs with nominal power P=25-1000 W have $\eta=7\div18$ lm/W

The fluorescent lamps have $\eta\approx50$ lm/W.

The efficiency in visible

$$\boxed{\eta_{viz} = \frac{P_{viz}}{P}\cdot 100 \;\; [\%]}$$

- the radiated power in visible (P_{vis})

- the total radiated power (P)

η_{viz} =3÷4 % for incandescent bulbs
η_{viz} =20 % for fluorescent lamps
η_{viz} approaches (theoretically) 60 % for LEDs

η_{vis} increases with the temperature increasing of the surface that emits e.m. radiation.

According to Wien's Displacement Law, v_{max}~T.

5.9.2. Early light sources

Incandescent Light Bulbs

Tungsten bulbs

On 13 December 1904, Hungarian Sándor Just and Croatian Franjo Hanaman were granted a Hungarian patent for a **tungsten (W)** filament lamp that lasted longer and gave brighter light than the carbon filament. Tungsten filament lamps were first marketed by the Hungarian company Tungsram in 1904.

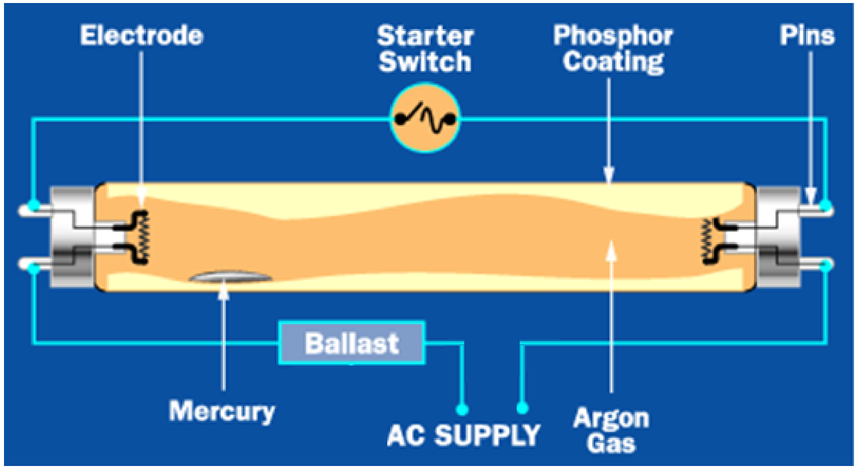
Original carbon-filament bulb from Thomas Edison; time life: 13.5 hours
Early carbon filaments had a negative temperature coefficient of resistance: as they got hotter, their electrical resistance decreased ➔ the lamp sensitive to fluctuations

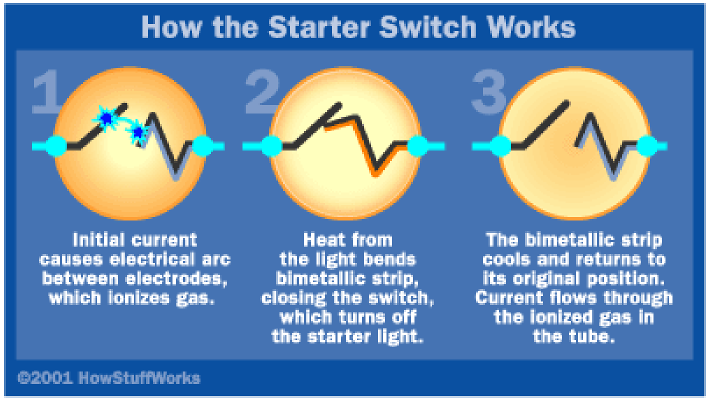
The bulb is filled with an inert gas such as argon (93%) and nitrogen (7%) to reduce evaporation of the filament and prevent its oxidation at a pressure of about 70 kPa (0.7 atm)

An electric current heats the filament to typically 2000 to 3300 K, well below tungsten's melting point of 3695 K.

Xenon halogen lamp

Fluorescent Lamp Operation





1 Initial current causes electrical arc between electrodes, which ionizes gas.

2 Heat from the light bends bimetallic strip, closing the switch, which turns off the starter light.

3 The bimetallic strip cools and returns to its original position. Current flows through the ionized gas in the tube.

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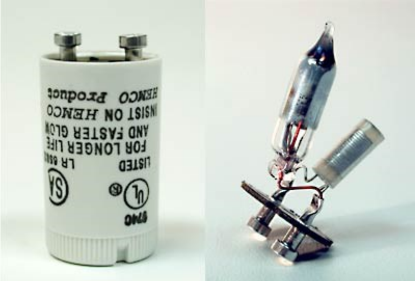
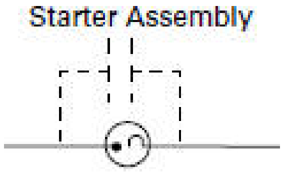
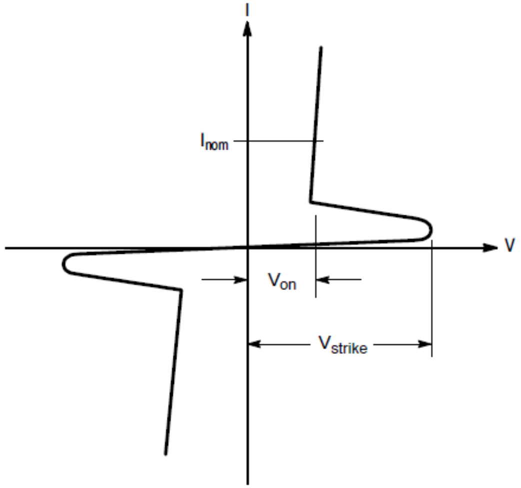
Schematic for "Ballast"

Starter







Typical low pressure fluorescent tube I/V characteristic

5.10. Electromagnetic radiation sources – basic aspects

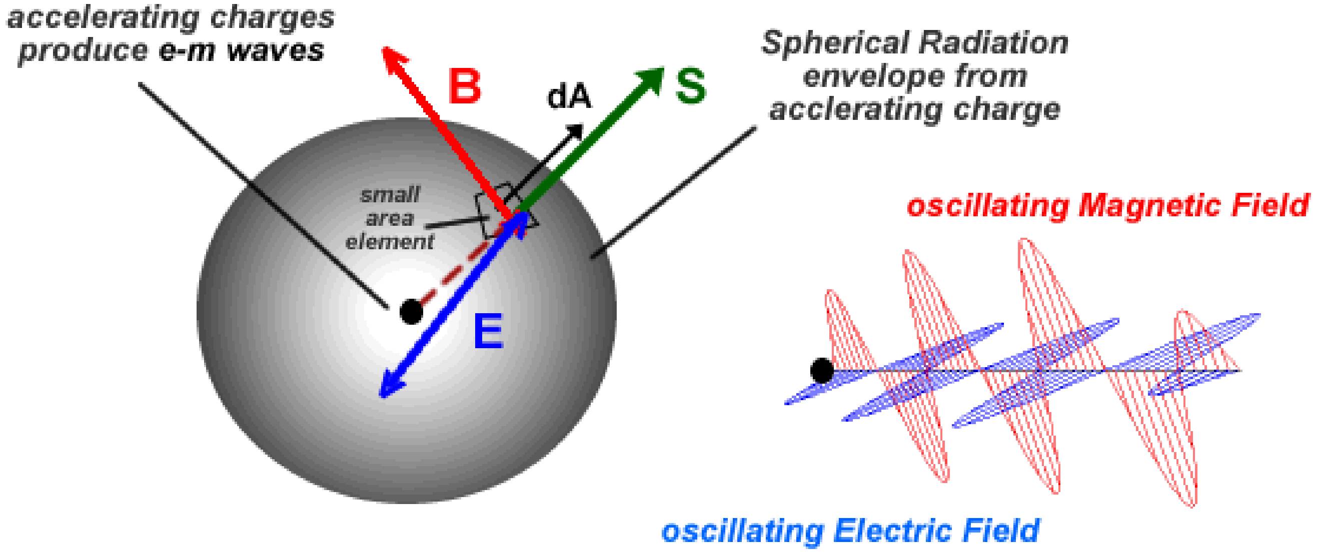
The image bellow describes in a suggestive way the generation of e.m. waves by *accelerating charges.*

Total Power Radiated

$$P = \oint S \cdot dA = \frac{1}{\mu_o} \oint (E \times B) \cdot dA$$

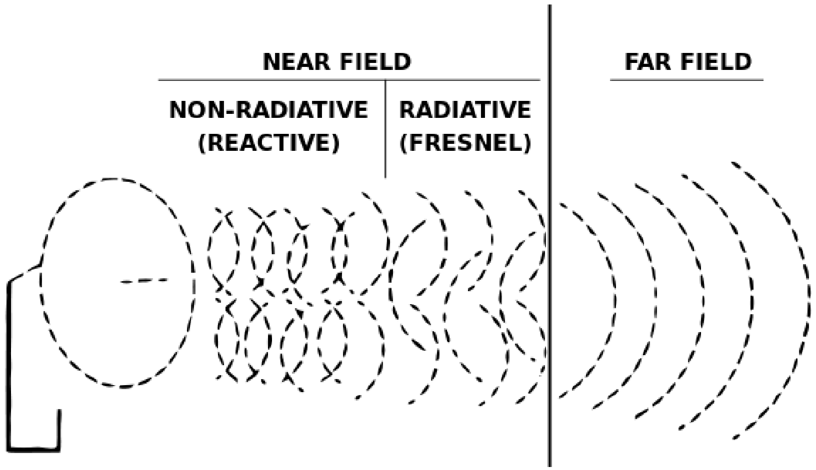
Energy Flux Density

$$S = \frac{1}{\mu_o} (E \times B)$$



Static sources do not radiate energy

5.10.1. Near field and far field

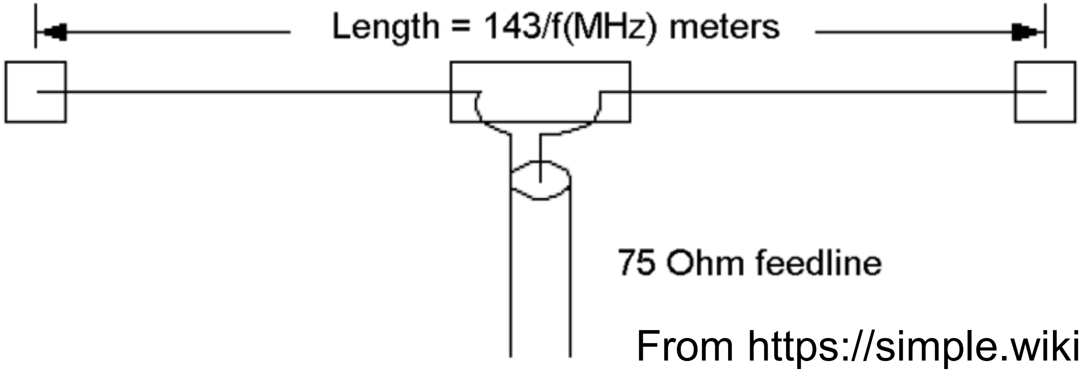


In electromagnetic radiation (such as microwaves from an antenna, shown above) the term "radiation" applies only to the parts of the electromagnetic field that radiate into infinite space and decrease in intensity by an inverse-square law of power, so that the total radiation energy that crosses through an imaginary spherical surface is the same, no matter how far away from the antenna the spherical surface is drawn.

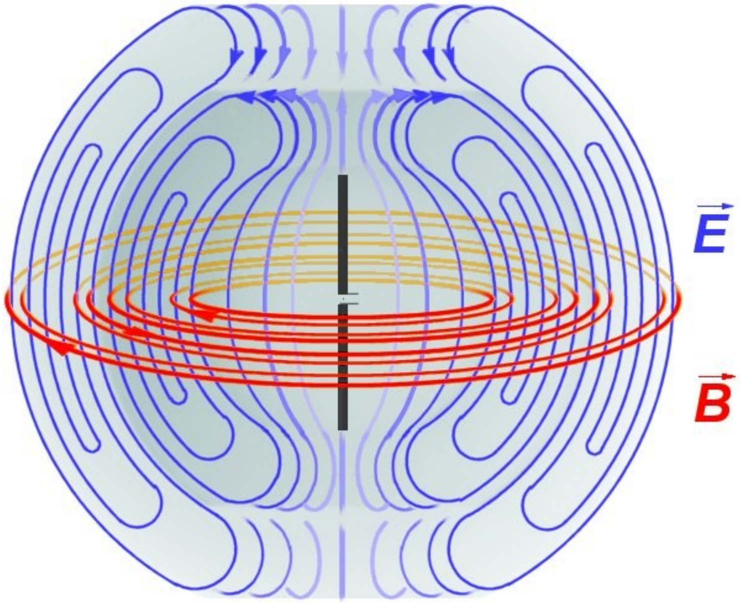
Electromagnetic radiation includes the far field part of the electromagnetic field around a transmitter and the "near-field" close to the transmitter, where a bi-directional exchange of energy between the electromagnetic field and antenna takes part and does not count as electromagnetic radiation.

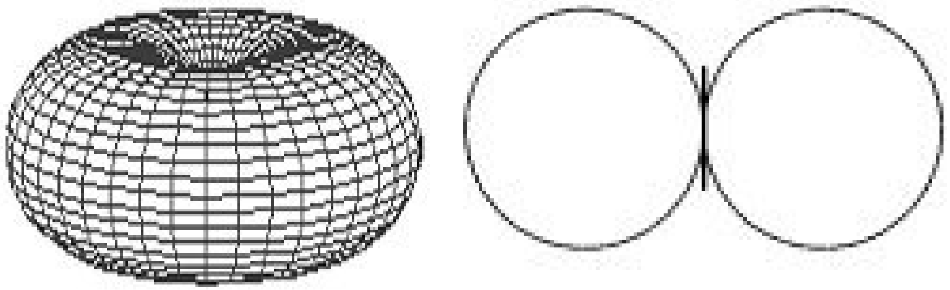
5.10.2. Dipole antenna

A simple *dipole antenna* and the generated $\vec{E}$ and $\vec{B}$ fields



From https://simple.wikipedia.org/wiki/Dipole_antenna





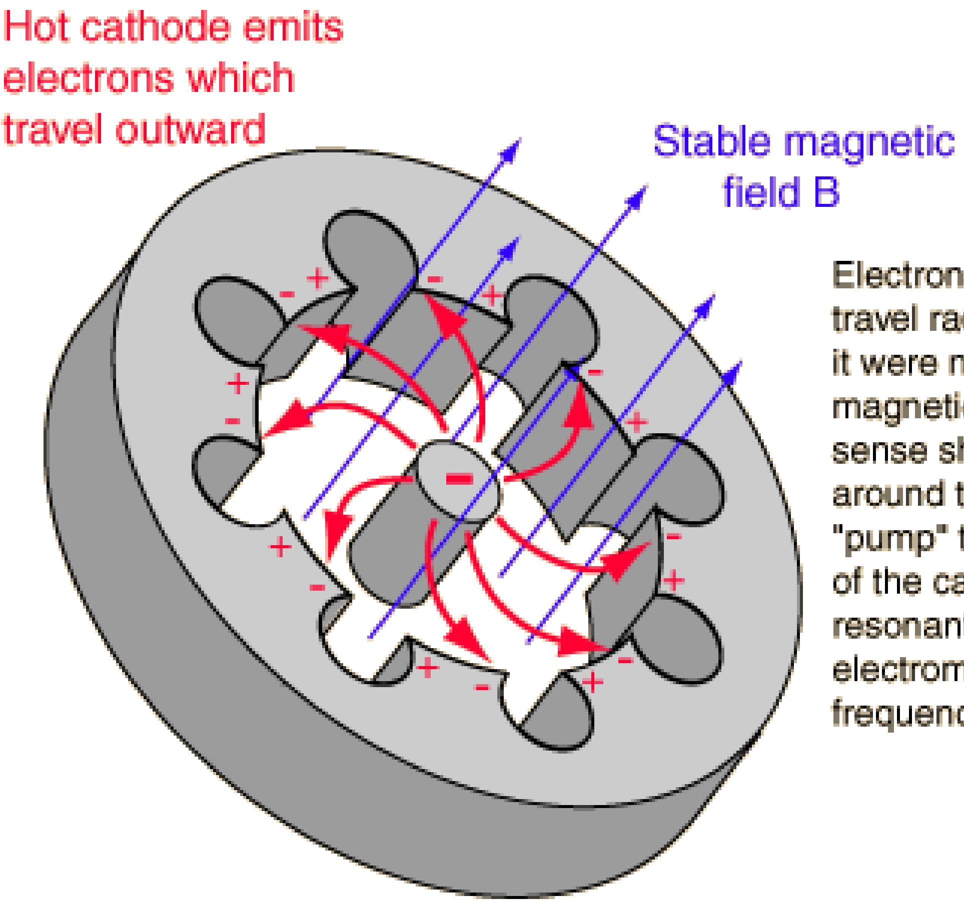
Three-dimensional perspective and the profile of the radiation pattern of an elementary doublet.

See https://demonstrations.wolfram.com/DipoleAntennaRadiationPattern/

5.10.3. Microwave sources

The magnetron - the microwave radiation of microwave ovens and some radar applications is produced by a device called **magnetron**.

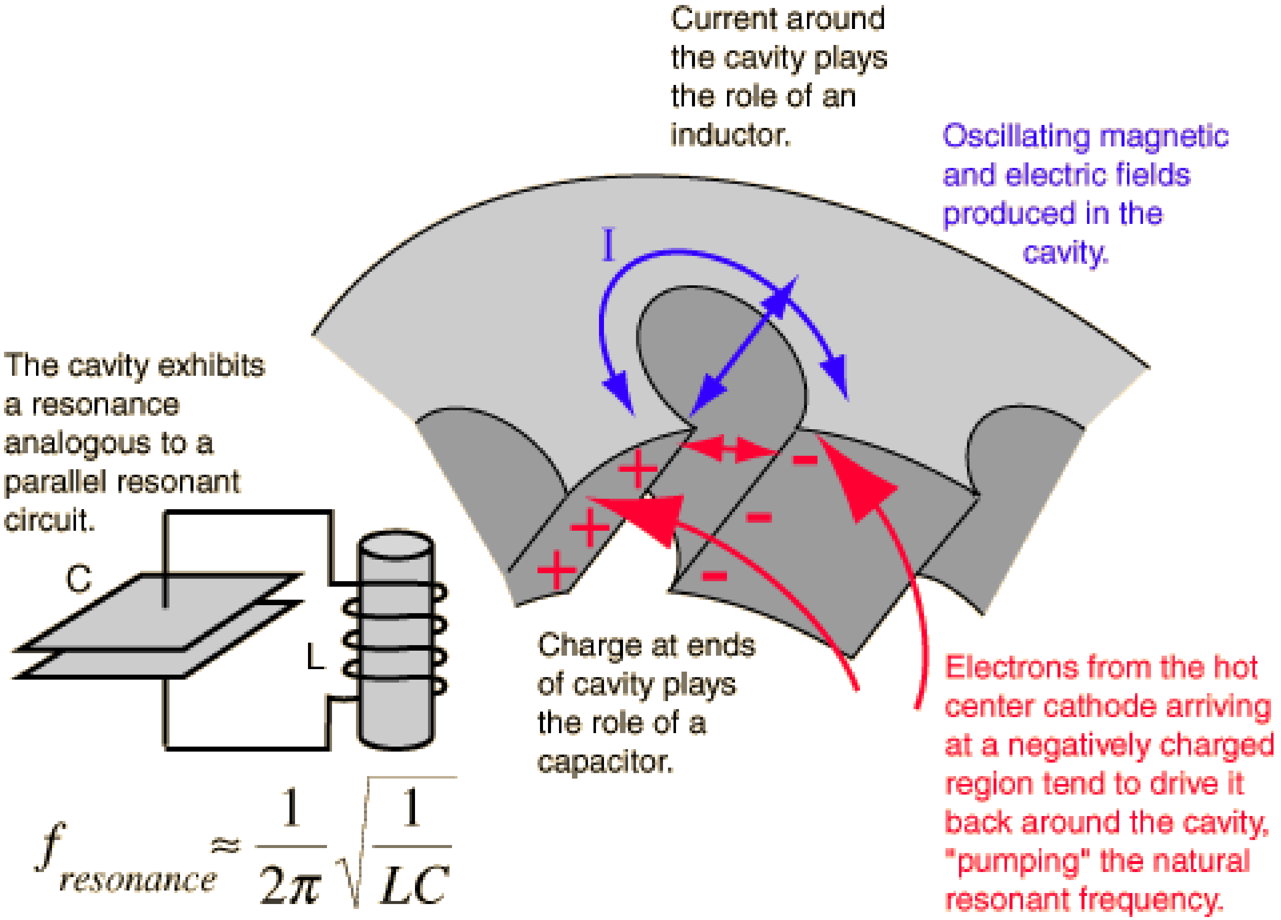
- this is a "crossed-field" device



Electrons from a hot filament would travel radially to the outside ring if it were not for the magnetic field. The magnetic force deflects them in the sense shown and they tend to sweep around the circle. In so doing, they "pump" the natural resonant frequency of the cavities. The currents around the resonant cavities cause them to radiate electromagnetic energy at that resonant frequency.

http://hyperphysics.phy-astr.gsu.edu/hbase/Waves/magnetron.html

Electrons are released at the center hot cathode by the process of thermionic emission. The axial magnetic field exerts a magnetic force on these charges - they tend to be swept around the circle. The system behaves like an LC oscillator.



Current around the cavity plays the role of an inductor.

Oscillating magnetic and electric fields produced in the cavity.

The cavity exhibits a resonance analogous to a parallel resonant circuit.

Charge at ends of cavity plays the role of a capacitor.

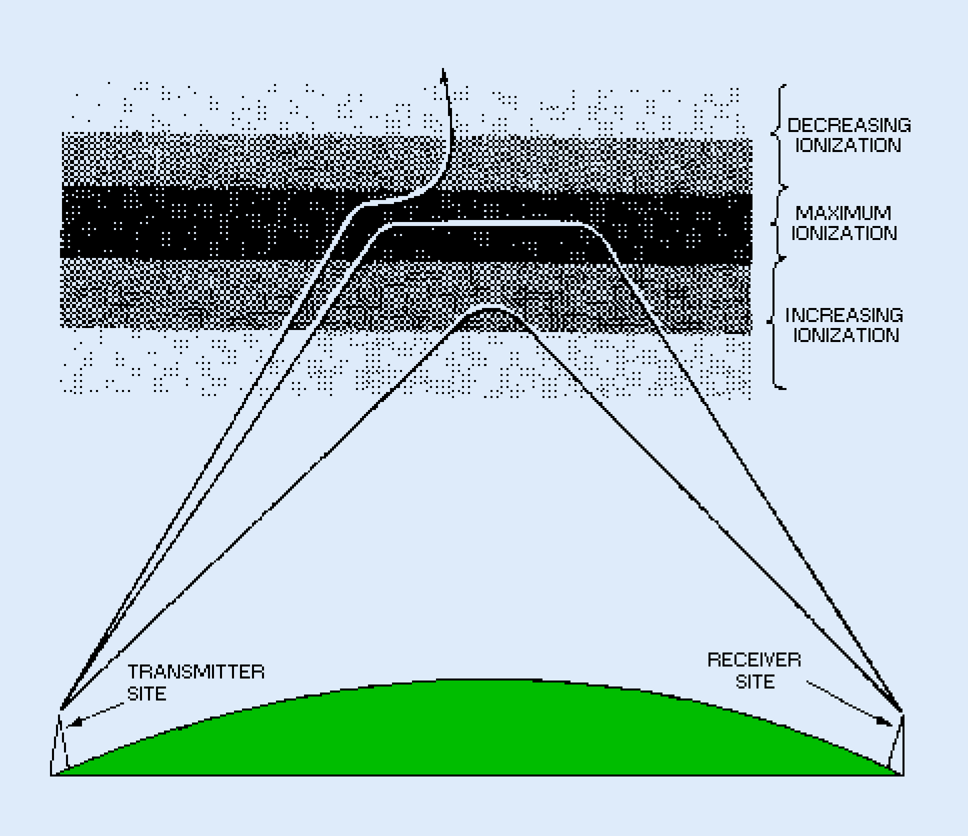
Electrons from the hot center cathode arriving at a negatively charged region tend to drive it back around the cavity, "pumping" the natural resonant frequency.

$$f_{resonance} \approx \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$$

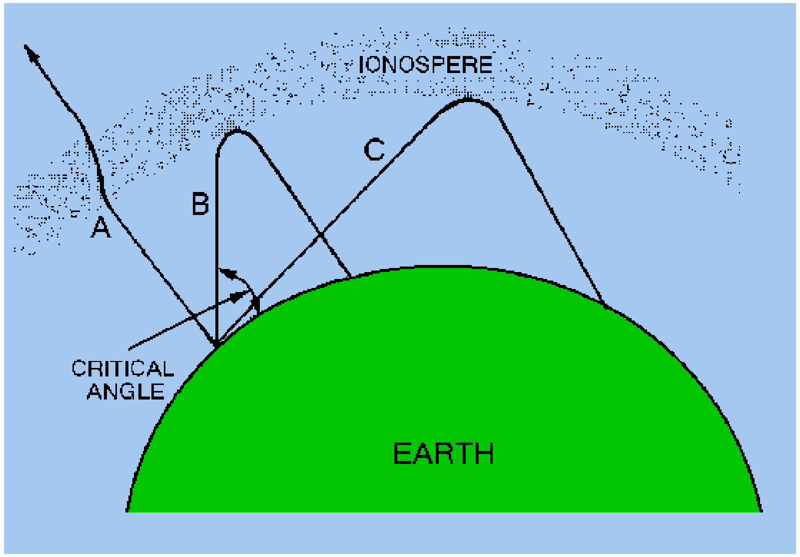
http://hyperphysics.phy-astr.gsu.edu/hbase/Waves/magnetron.html

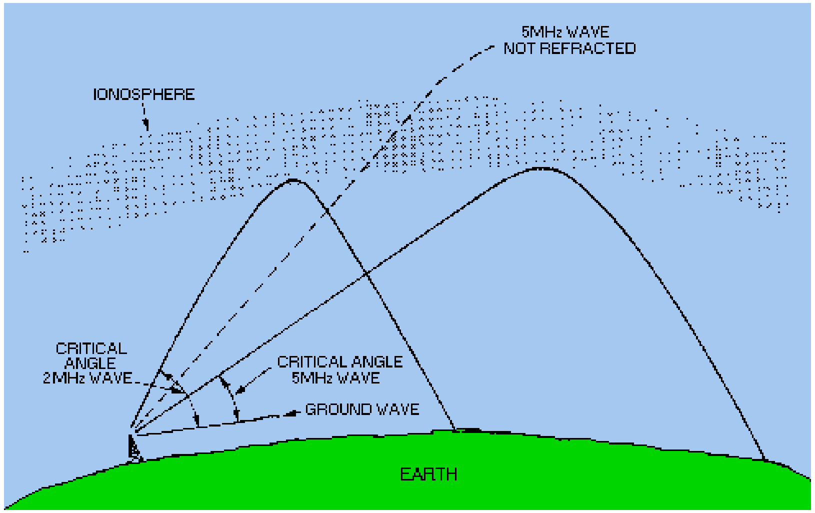
5.10.4. Ionospheric reflection – basic aspects

Ionospheric reflection is a bending, through a complex process involving reflection and refraction, of electromagnetic waves propagating in the ionosphere back toward the Earth. The amount of bending depends on the extent of penetration (which is a function of frequency), the angle of incidence, polarization of the wave, and ionospheric conditions, such as the ionization density. It is negatively affected by incidents of ionospheric absorption.



Effects of ionospheric density on radio waves





Different incident angles of radio waves

Chapter VI. Waves polarization

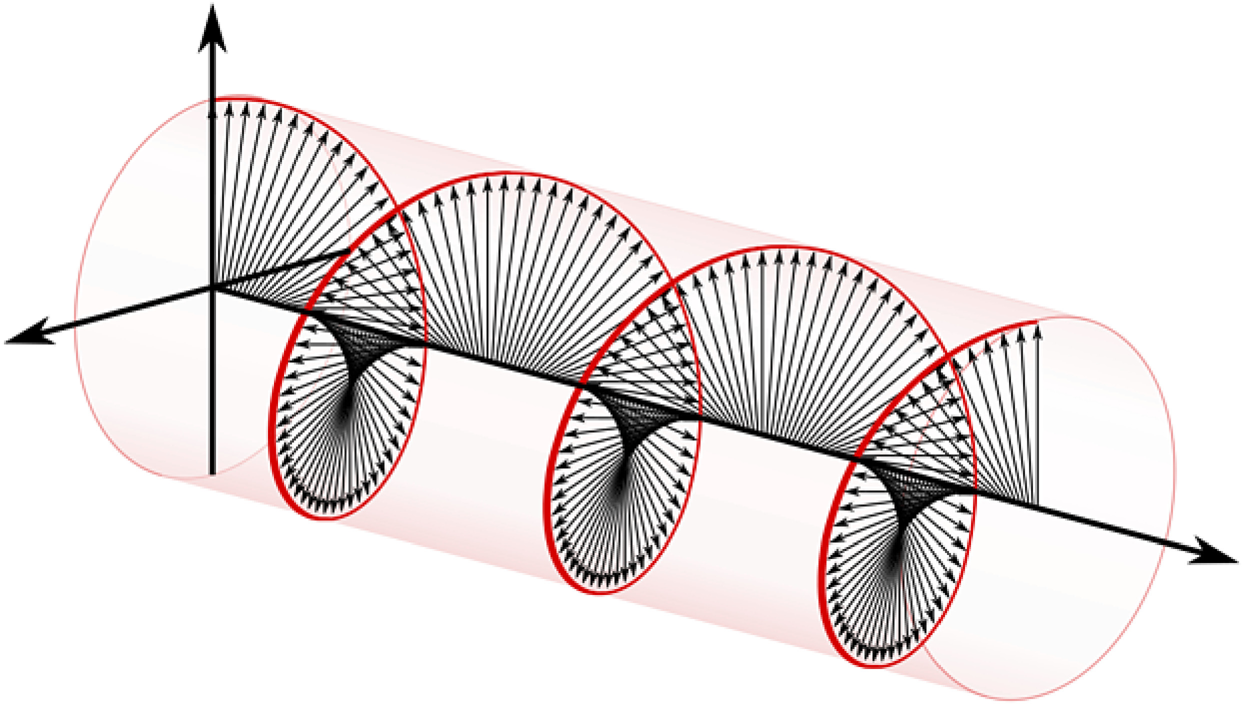


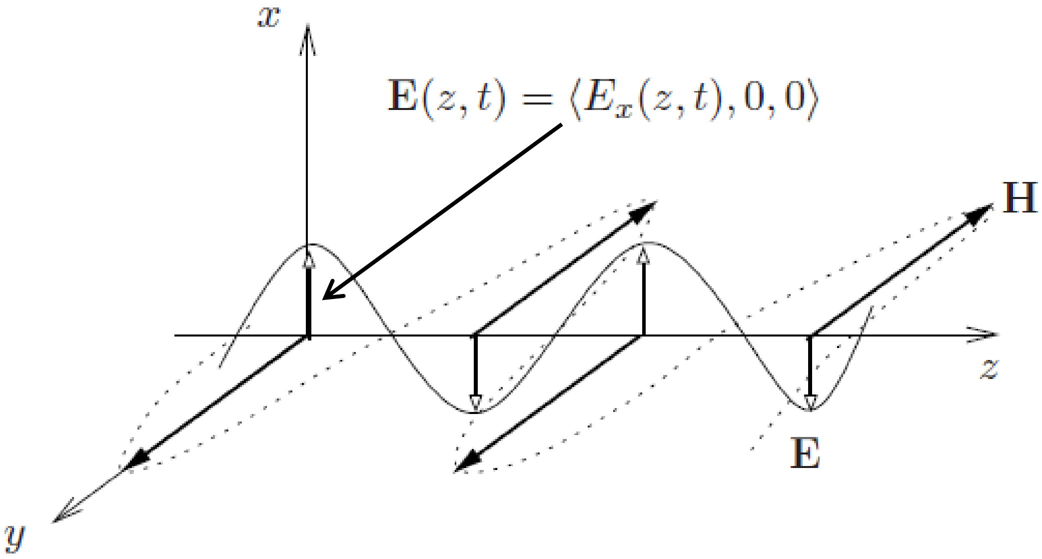
Image from https://www.rfvenue.com/blog/2014/12/15/wave-polarization-explained

Extended Table of Contents

6.1. Introduction

The polarization of a plane wave refers to the orientation of the E-field vectors in the plane perpendicular to the direction of propagation.

Up till now, we have considered only the simplest case known as **linear polarization**, in which the E field lines are orientated at a fixed angle in the plane.



$$\vec{H} \perp \vec{n}, \vec{E} \perp \vec{n}$$

Sinusoidal e.m. wave propagating in the z direction

Remember that: $\boxed{\dfrac{E}{H} = \sqrt{\dfrac{\mu}{\varepsilon}}} \Rightarrow \boxed{H = E\dfrac{1}{\sqrt{\dfrac{\mu}{\varepsilon}}} = E\dfrac{1}{\eta}}$ η – the characteristic impedance

So, we can define:

$$\boxed{\eta_0 = \frac{E}{H} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 377\ \Omega}$$

- the characteristic impedance in free space

$$\boxed{\eta = \frac{E}{H} = \sqrt{\frac{\mu_r \mu_0}{\varepsilon_r \varepsilon_0}} \approx 170\ \Omega}$$

- the characteristic impedance of a dielectric medium;
- in glass, $\eta \approx 170\ \Omega$

We consider the following waves equations for E and H fields that propagate over z axis.

$$E_x(z,t) = E_0 \cos(\omega \cdot t - k \cdot z + \psi)$$

$$H_y(z,t) = \vec{H}_0 \cos(\omega \cdot t - k \cdot z + \psi) = \frac{1}{\eta} E_x$$

or

$$\vec{E} = \vec{E}_0 e^{i(\omega t - kz + \psi)}$$

$$\vec{H} = \vec{H}_0 e^{i(\omega t - kz + \psi)}$$

More generally, a forward travelling sinusoidal EM wave travelling in the +z direction can be represented as the sum of two orthogonal components

$$\vec{E}(z) = \vec{i}\,E_1 e^{-ikz} + \vec{j}\,E_2 e^{i\psi} e^{-ikz}$$

- E_1 and E_2 are the complex amplitudes of the x and y components
- ψ represents a possible relative phase shift between them

The associated H field is given by

$$\vec{H}(z) = -\vec{i}\,\frac{1}{\eta} E_2 e^{i\psi} e^{-ikz} + \vec{j}\,\frac{1}{\eta} E_1 e^{-ikz}$$

The physical sinusoidal electric and magnetic fields are modelled by the pair

$$\vec{E}(z,t) = \vec{i}\,E_1 \cos(\omega \cdot t - k \cdot z) + \vec{j}\,E_2 \cos(\omega \cdot t - k \cdot z + \psi)$$

$$\vec{H}(z,t) = -\vec{i}\,\frac{1}{\eta} E_2 \cos(\omega \cdot t - k \cdot z + \psi) + \vec{j}\,\frac{1}{\eta} E_1 \cos(\omega \cdot t - k \cdot z)$$

There are three classes of polarization, which depend on the relative amplitudes of E_1 and E_2, and the phase shift ψ between them.

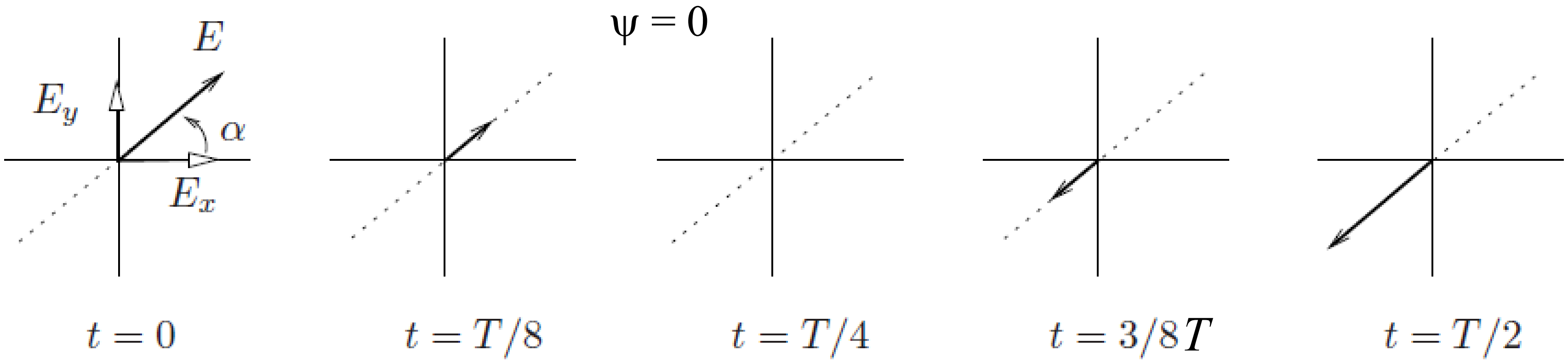
6.2. Types of wave polarization

6.2.1. Linear polarization

E_x and E_y are in phase i.e. $\psi = 0$

The electric vector is orientated at an angle determined by the relative values of amplitudes E_1 and E_2.

The amplitude of the vector varies sinusoidally, illustrated by a sequence of snapshots in a fixed plane (e.g. at z = 0), at different fractions of the period T of the sinusoidal wave.

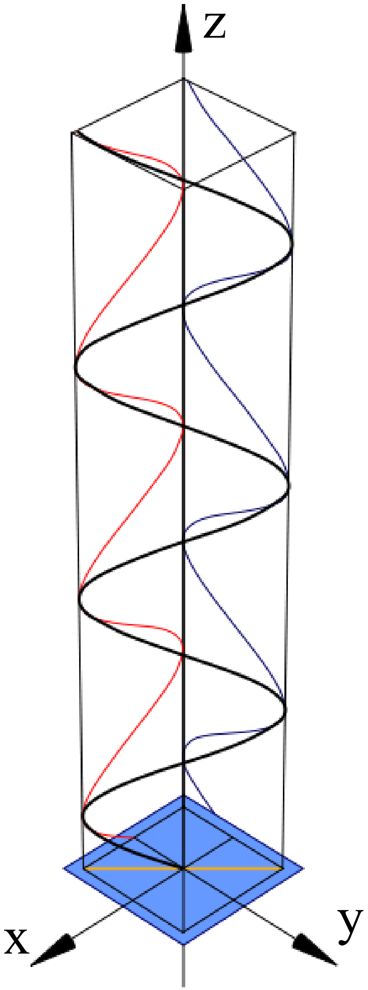


Linear polarization – snapshots in time

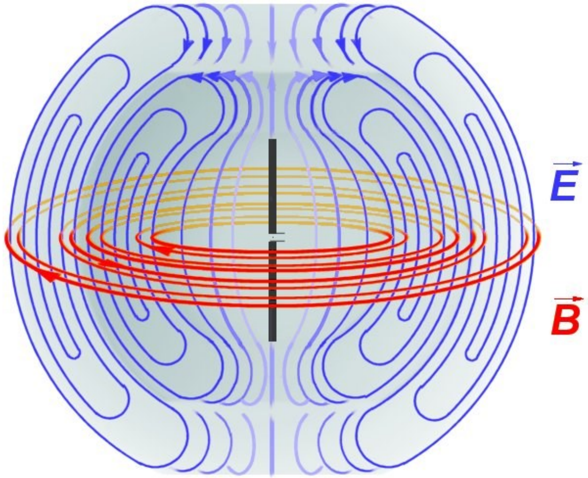
$$\sqrt{E_x^2 + E_y^2} = \sqrt{E_1^2 \cos^2(\omega t - kz) + E_2^2 \cos^2(\omega t - kz)} = \sqrt{E_1^2 + E_2^2}\,|\cos(\omega t - kz)|$$

$$\sqrt{H_x^2 + H_y^2} = \sqrt{\left(\frac{-E_y}{\eta}\right)^2 + \left(\frac{E_x}{\eta}\right)^2} = \frac{1}{\eta}E \qquad tg\alpha = \frac{E_y}{E_x} = \frac{E_2}{E_1}$$

The orientation angle α jumps 180 degrees, when the resultant passes through zero



- the evolution of the electric field vector (black), with time (the vertical axes), at a particular point in space, along with its *x* (red) and *y* (blue) components; at the base is the path traced by the vector in the transverse plane.



$\vec{E}$ and $\vec{B}$ fields generated by a simple dipole Radiating antenna

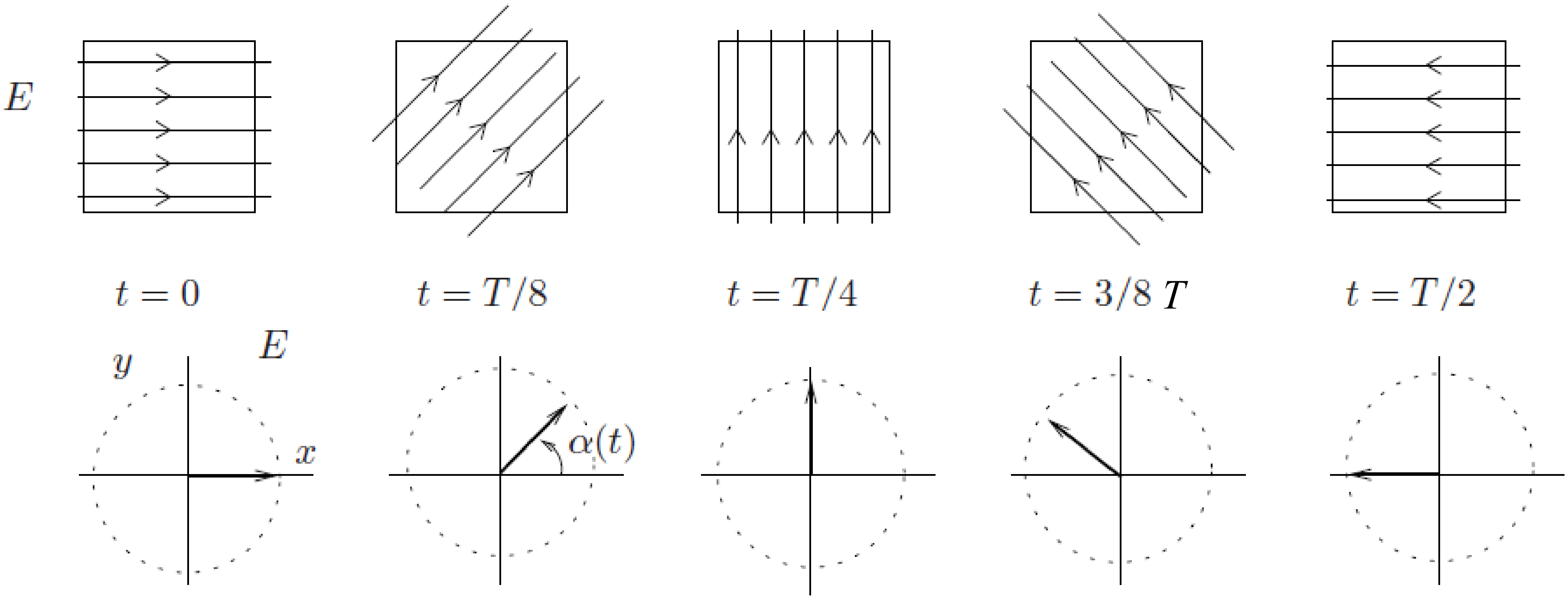
It is common in telecommunications and radar engineering to describe the polarization by the orientation of the electric field vector: "vertical polarization" if the electric field is orientated vertically, and "horizontal polarization" if the electric field is horizontal. The orientation of the polarization is dependent on the orientation of the radiating antenna.

6.2.2. Circular polarization

$E_1 = E_2$ and $\psi = \pm \pi/2$

If the two orthogonal components are equal in amplitude, but with a relative phase shift of 90 degrees, **the total electric field vector** will "rotate" as a function of time if viewed in a plane perpendicular to the direction of propagation at a fixed location in space.

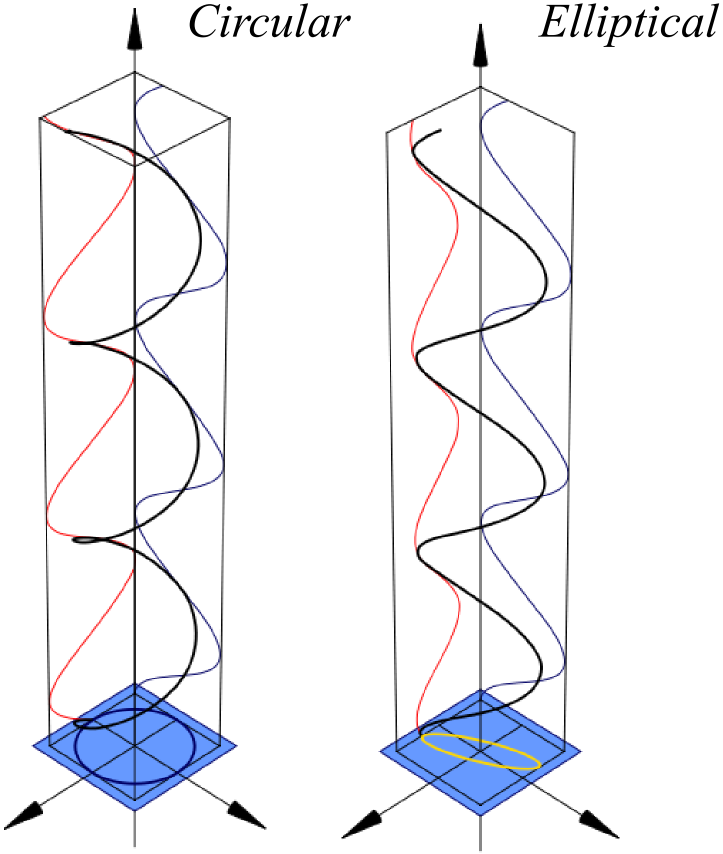
Circular Polarization - snapshots in time.



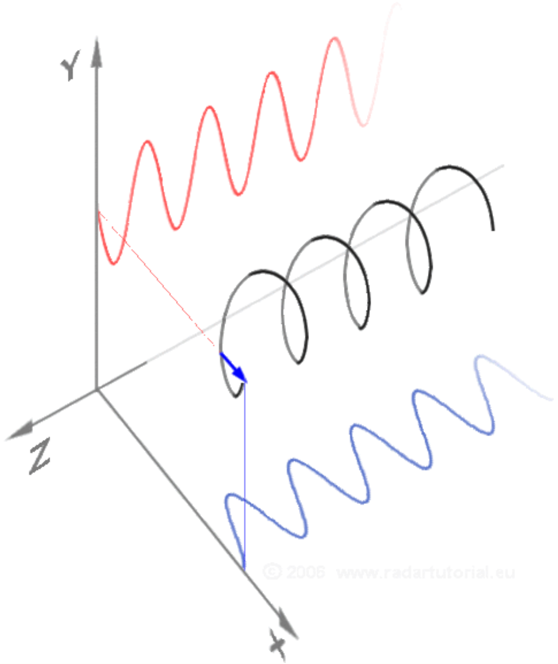
The electric field lines contained in the *xy* plane will rotate as a function of time at a rate of ω rad/s, but the magnitude, indicated by the line spacing, does not change.

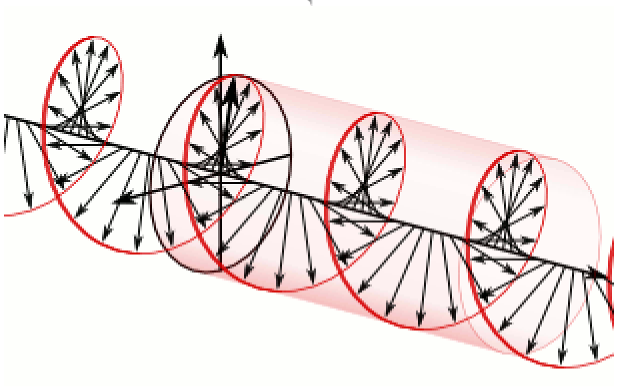
Representation of the electric field vector of a wave of circularly polarized electromagnetic radiation.

Note, this is “right-hand” circular polarization.



- the vectors wind around like a “corkscrew”





By Dave3457 - Own work, Public Domain, https://commons.wikimedia.org/w/index.php?curid=9863231

To analyze the case of circular polarization, we set $E_1 = E_2 = A$ and $\psi = \pm \pi/2$.

The magnitude of the resultant E-vector is given by:

$$\sqrt{E_x^2 + E_y^2} = \sqrt{A^2 \cos^2(\omega t - kz) + A^2 \sin^2(\omega t - kz)} = A$$

- is independent of z and t.

The orientation angle is however time-varying, and may be calculated from:

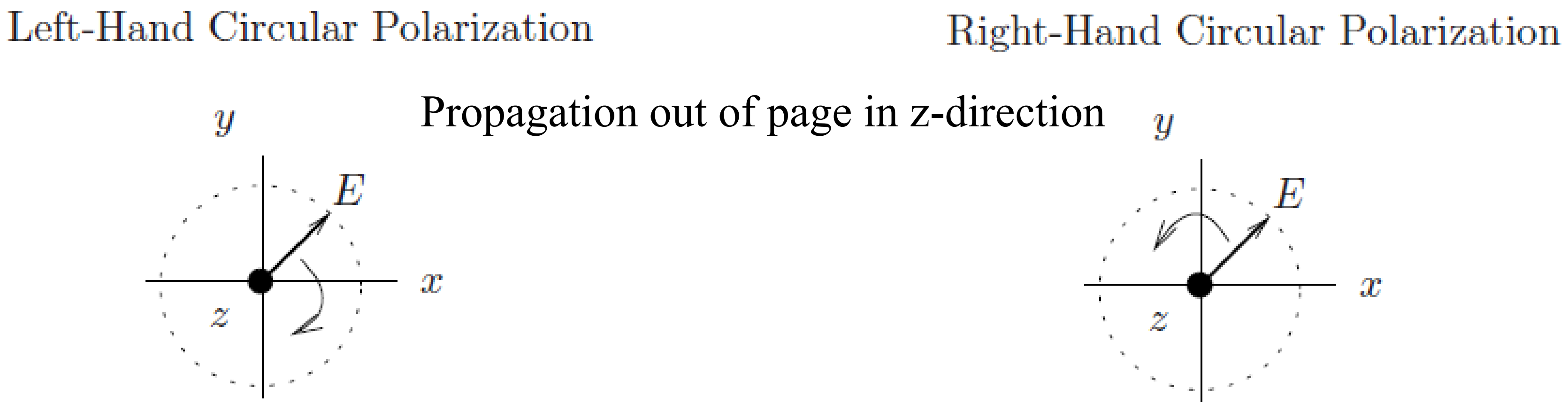
$$tg\alpha = \frac{E_y}{E_x} = \frac{A\cos(\omega t - kz \pm \pi/2)}{A\cos(\omega t - kz)} = \frac{\mp A\sin(\omega t - kz)}{A\cos(\omega t - kz)} = \overset{\Psi = \frac{\pi}{2}}{\underset{\Psi = -\frac{\pi}{2}}{\mp}} tg(\omega t - kz)$$

NOTE: At a fixed position z, the vector rotates at a rate of ω radians per second in a circle in the plane. At a fixed time t, the vector rotates at k radians per metre

The IEEE defines two types of circular polarization, according to the direction of rotation:

Left-hand circular: The E vector of a circularly polarized plane wave propagating out of the x-y plane rotates in the clockwise direction in the x-y plane.
It is noted that in our particular example, this corresponds to the case where $\psi = \pi/2$.

Right-hand circular – The E vector of a circularly polarized plane wave propagating out of the x-y plane rotates in the counter clockwise direction in the x-y plane.
In our particular example, this corresponds to the case where $\psi = -\pi/2$.


Left-Hand Circular Polarization

Right-Hand Circular Polarization

Propagation out of page in z-direction

Left-hand versus right-hand polarization.

One can easily plot the E vector in the x−y plane for t = 0, T/8, 2T/8, 3T/8, 4T/8 by considering the E_x and E_y components in our polarization model

$$\vec{E}(z,t)=\vec{i}\,E_1\cos(\omega\cdot t-k\cdot z)+\vec{j}\,E_2\cos(\omega\cdot t-k\cdot z+\psi)$$

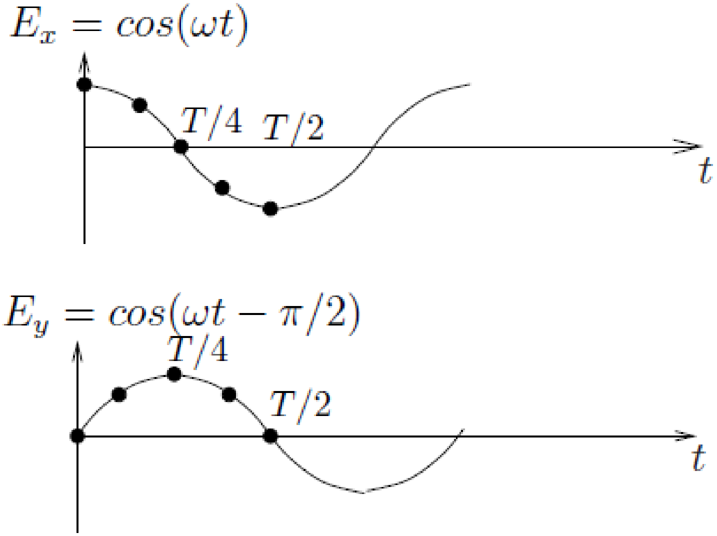
$$\Psi=-\frac{\pi}{2}$$

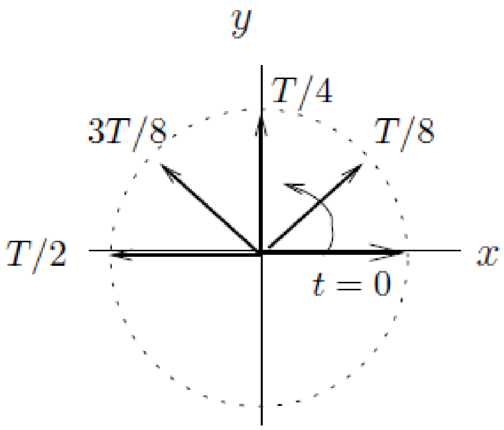
$$z=0$$

$$E_x=A\cos(\omega t)$$

$$E_y=A\cos(\omega t-\frac{\pi}{2})$$

The E_x and E_y values are plotted as functions of time below:





E vector rotates in $z = 0$ plane

Circular Polarization - snapshots in time (at a fixed position, z=0)

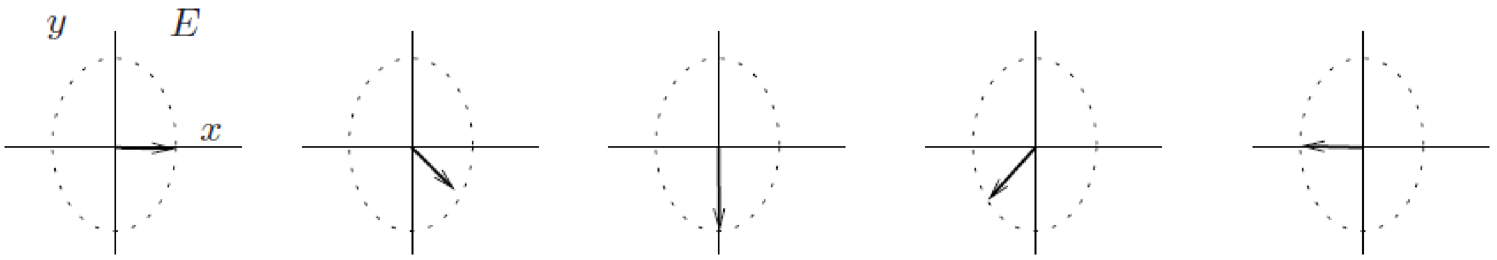
Wave propagates in +z direction out of page; hence this is right hand circular polarization

6.2.3. Elliptical polarization

Either $E_1 \neq E_2$ or $\psi \neq \{0, \pi/2, -\pi/2\}$

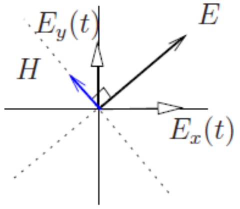
Elliptical polarization is the general case, in which the vector **not only rotates**, but also **varies in length**, tracing out an ellipse in a plane at a fixed position, as illustrated in the sequence below (which is left-hand elliptical polarization):

Elliptical Polarization - snapshots in time.

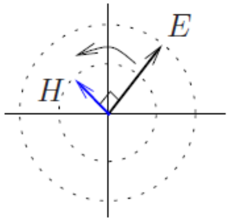


$t = 0$ $t = T/8$ $t = T/4$ $t = 3/8\,T$ $t = T/2$

COMMENT: In all three classes of polarization, the H-field is always perpendicular to the E-field, and in phase with it:

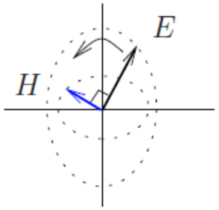


Linear Polarization
- fixed orientation of resultant
- amplitude varies sinusoidally.



Circular Polarization

- rotating vector traces a circle



Elliptical Polarization

- rotating vector traces an ellipse

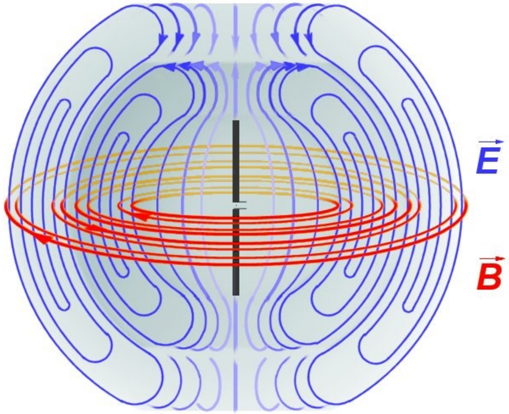
6.3. Applications of wave polarization

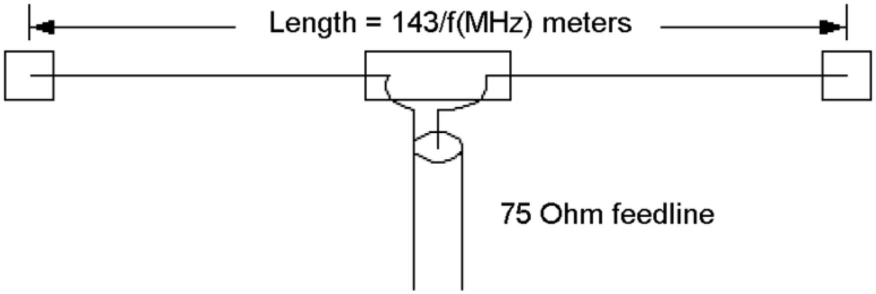
6.3.1. Communication links

In broadcast transmitters e.g. radio, TV signal, **linear polarization** is most commonly used. Wire antennas like dipoles, monopole and Yagi antennas radiate linearly polarized radiation. Both **horizontal** and **vertical polarization** is used.

The orientation of the transmitting antenna determines the orientation of the polarization. The receiving antenna must be orientated correctly to receive the maximum signal strength.

Ex.: a vertically polarized EM wave ➔ The dipole must be orientated vertically to receive the signal – the electrons rush vertically up and down the wires in response to the incoming E-field, resulting in a "voltage response" across the centre terminals.
At an arbitrary angle, the signal drops off as the cosine of the angle from the vertical (i.e. with the component of the electric field in the direction of the dipole rod).





6.3.2. Radar - “Radio detection and ranging”

Radar is a technique used to detect far away metallic targets like airplanes, …

A pulse is transmitted in the direction of interest, and the return echo is received and used to detect the presence of targets.
Circular polarization is sometimes used in radar applications as some scattering structures only reflect electromagnetic energy of a particular polarization. In circular polarization, the E-field vector rotates in a plane at the reflector (being a fixed distance from the source), which ensures that there will always be some reflection from such structures.
For example, a vertically orientated thin wire rod will only reflect vertical polarization, where as a horizontally orientated wire rod will only reflect horizontal polarization.
Circular polarization can be thought of as the sum of two orthogonal, linearly polarized components. On reflection, only one component, in line with the orientation of the reflecting rod will be reflected.

⇩

An incident circularly polarized wave will be linearly polarized on reflection.

Circular polarization can be generated by using two perpendicular dipoles (crossed-dipoles), driven 90 degrees out of phase. Alternatively, the perpendicular dipoles may be driven in phase, but physically separated by a quarter wavelength in the propagation direction to achieve the required 90 degree phase shift.

6.3.3. Polarized 3D system

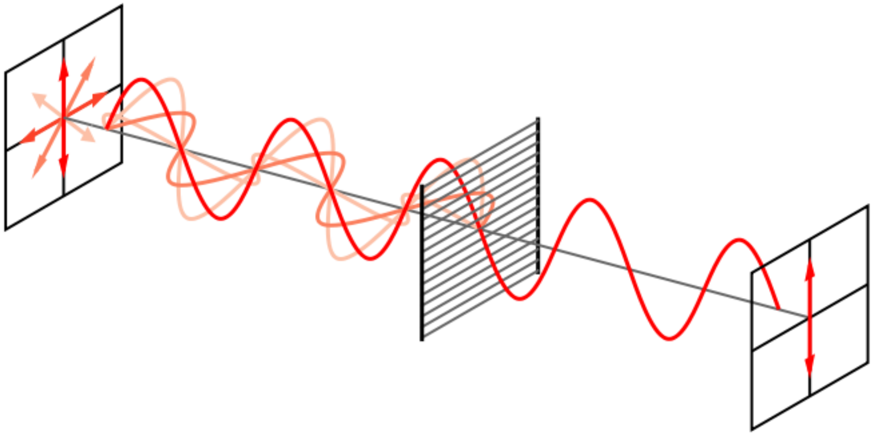
A polarized 3D system uses polarization glasses to create the illusion of three-dimensional images by restricting the light that reaches each eye - an example of stereoscopy.

To present stereoscopic images and films, two images are projected superimposed onto the same screen or display through different polarizing filters. The viewer wears low-cost eyeglasses which contain a pair of different polarizing filters. As each filter passes only that light which is similarly polarized and blocks the light polarized in the opposite direction, each eye sees a different image. This is used to produce a three-dimensional effect by projecting the same scene into both eyes, but depicted from slightly different perspectives.

Experiment:
Try to rotate circularly polarized 3D glasses in front of an LCD (Liquid Crystal Display) screen. See what is the effect of this rotation and try to give explanation based on the phenomena presented in these sections.

a) Linearly polarized glasses

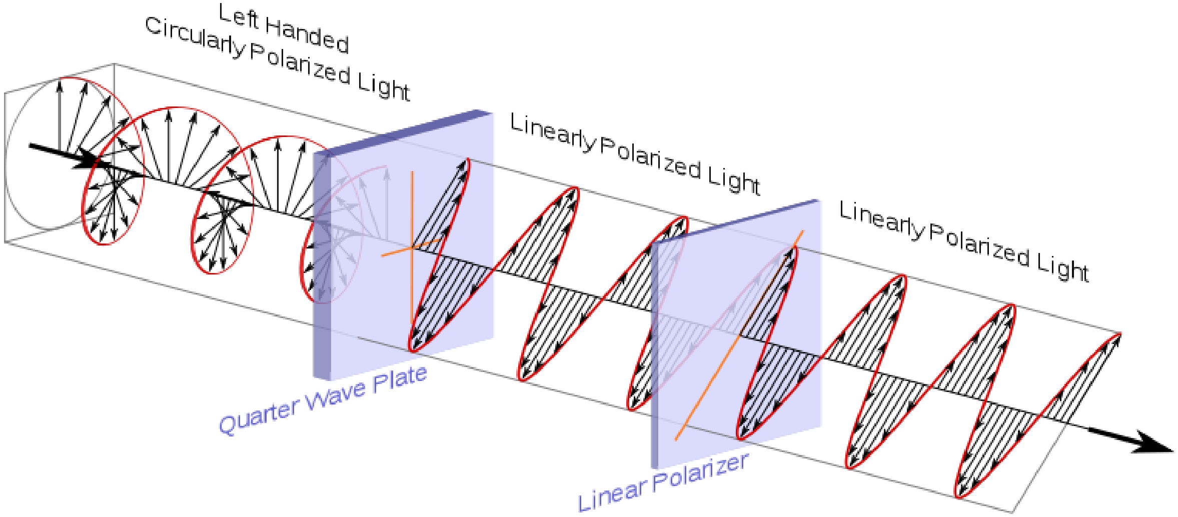
To present a stereoscopic motion picture, two images are projected superimposed onto the same screen through orthogonal polarizing filters (Usually at 45 and 135 degrees).
The viewer wears linearly polarized eyeglasses which also contain a pair of orthogonal polarizing filters oriented the same as the projector. As each filter only passes light which is similarly polarized and blocks the orthogonally polarized light, each eye only sees one of the projected images, and the 3D effect is achieved. Linearly polarized glasses require the viewer to keep his or her head level, as tilting of the viewing filters will cause the images of the left and right channels to bleed over to the opposite channel. This can make prolonged viewing uncomfortable as head movement is limited to maintain the 3D effect.



A linear polarizer converts an unpolarized beam into one with a single linear polarization. The vertical components of all waves are transmitted, while the horizontal components are absorbed and reflected. Image from https://en.wikipedia.org/wiki/Polarized_3D_system

b) Circularly polarized glasses

To present a stereoscopic motion picture, two images are projected superimposed onto the same screen through circular polarizing filters of opposite handedness. The viewer wears eyeglasses which contain a pair of analyzing filters (circular polarizers mounted in reverse) of opposite handedness. Light that is left-circularly polarized is blocked by the right-handed analyzer, while right-circularly polarized light is extinguished by the left-handed analyzer. The result is similar to that of steroscopic viewing using linearly polarized glasses, except the viewer can tilt his or her head and still maintain left/right separation (although stereoscopic image fusion will be lost due to the mismatch between the eye plane and the original camera plane).



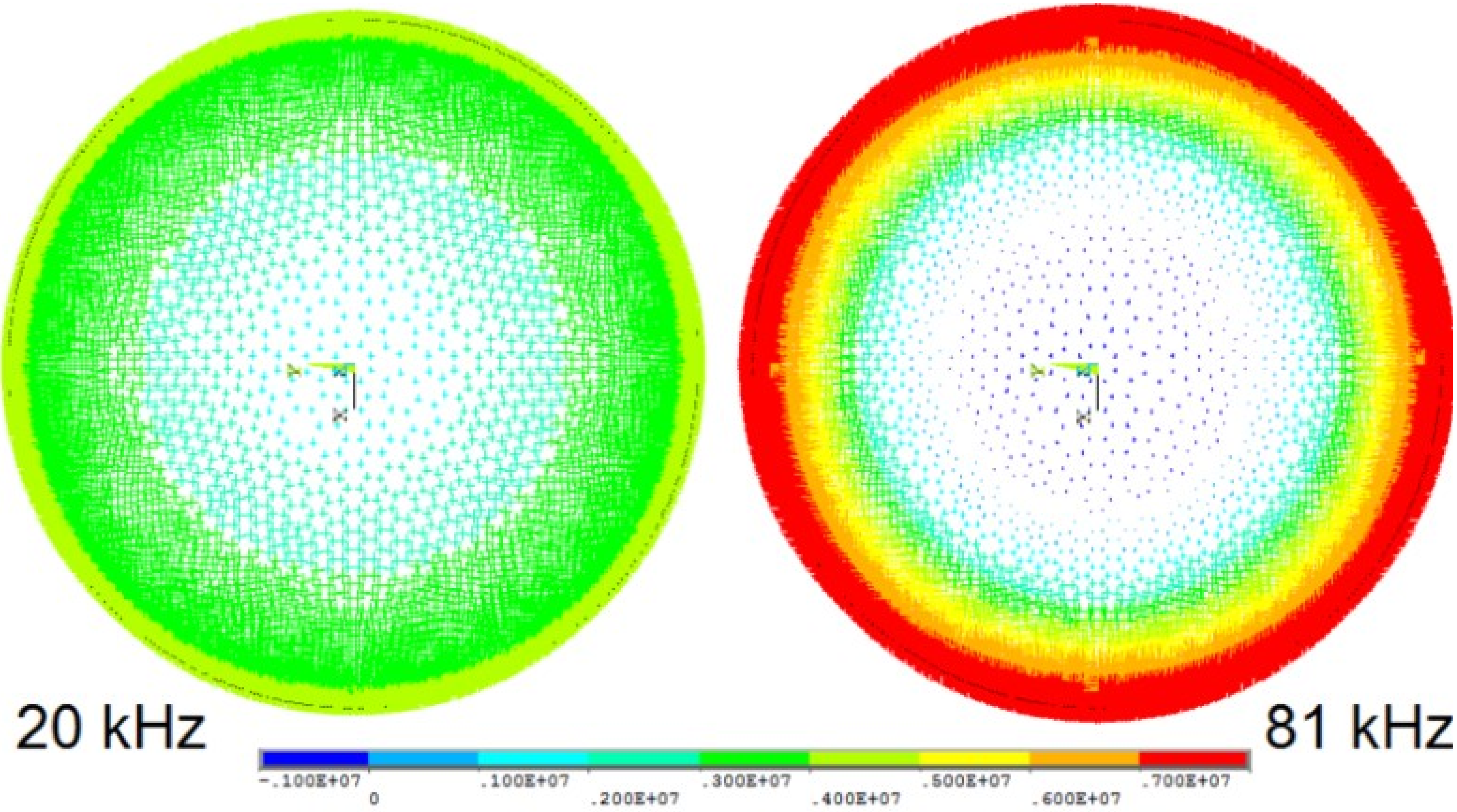
Circular polarizer passing left-handed, counter-clockwise circularly polarized light
Image from https://en.wikipedia.org/wiki/Polarized_3D_system

Some technical aspects:

The analyzing filters are constructed of a quarter-wave plate (QWP) and a linearly polarized filter (LPF). The QWP always transforms circularly polarized light into linearly polarized light. However, the angle of polarization of the linearly polarized light produced by a QWP depends on the handedness of the circularly polarized light entering the QWP. In the illustration, the left-handed circularly polarized light entering the analyzing filter is transformed by the QWP into linearly polarized light which has its direction of polarization along the transmission axis of the LPF. Therefore, in this case the light passes through the LPF. In contrast, right-handed circularly polarized light would have been transformed into linearly polarized light that had its direction of polarization along the absorbing axis of the LPF, which is at right angles to the transmission axis, and it would have therefore been blocked.

By rotating either the QWP or the LPF by 90 degrees about an axis perpendicular to its surface (i.e. parallel to the direction of propagation of the light wave), one may build an analyzing filter which blocks left-handed, rather than right-handed circularly polarized light. Interestingly, rotating both the QWP and the LPF by the same angle does not change the behaviour of the analyzing filter – **can you explain why?**.

Chapter VII. The skin effect



Current distribution inside of a conductor
Image from https://blog.solidsignal.com/tutorials/what-is-the-skin-effect/

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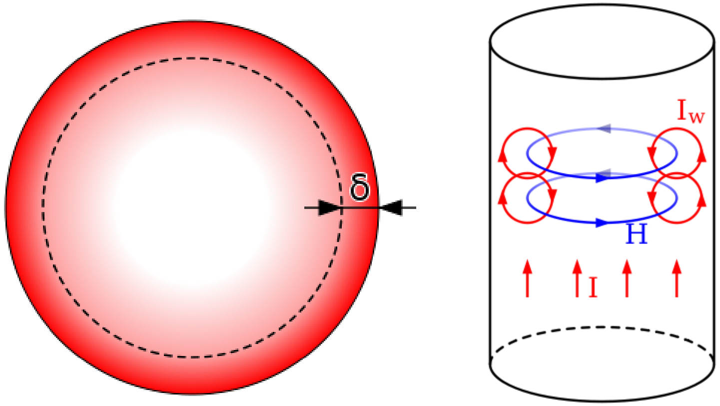
7.1. Introduction

Skin effect is the tendency of an alternating electric current (AC) to become distributed within a conductor such that the current density is largest near the surface of the conductor, and decreases with greater depths in the conductor. ***The electric current flows mainly at the "skin" of the conductor, between the outer surface and a level called the skin depth, δ*.** The skin effect causes the effective resistance of the conductor to increase at higher frequencies where the skin depth is smaller, thus reducing the effective cross-section of the conductor. The skin effect is due to opposing eddy currents induced by the changing magnetic field resulting from the alternating current. At 50 Hz in copper, the skin depth is about 9.2 mm!!. At high frequencies the skin depth becomes much smaller. Increased AC resistance due to the skin effect can be mitigated by using **specially woven litz wire**. Because the interior of a large conductor carries so little of the current, tubular conductors such as pipe can be used to save weight and cost.



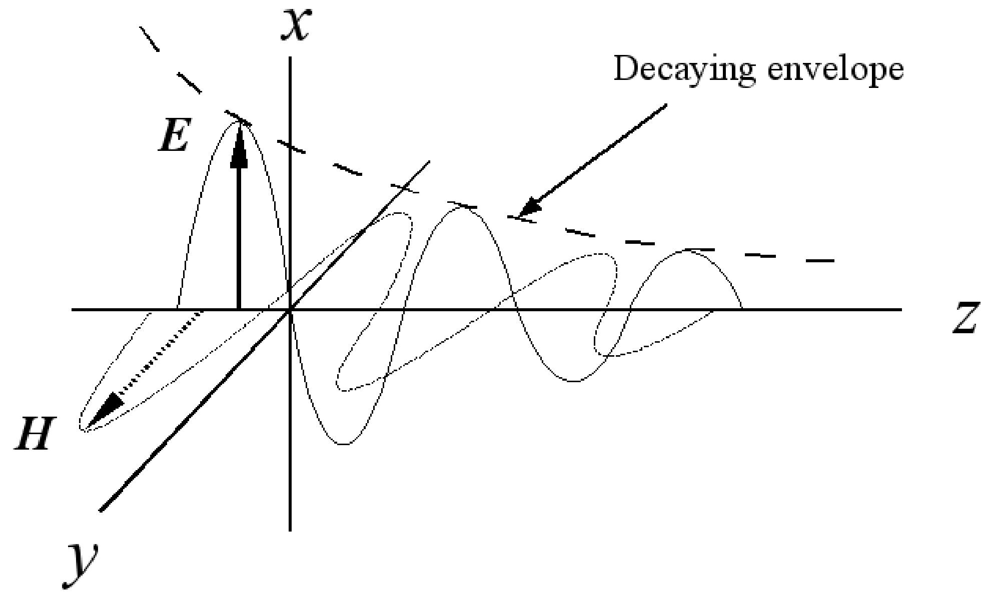
Skin depth is due to the circulating eddy currents (arising from a changing H field) cancelling the current flow in the center of a conductor and reinforcing it in the skin.

Image from https://en.wikipedia.org/wiki/Skin_effect

7.2. Wave propagation in conducting media and the skin depth

Wave propagation in a conducting medium

Wave propagation in a conducting medium results in an exponential decay of the wave amplitude as it propagates.



Exponentially decaying plane wave in a conducting (lossy) medium. **Note $\vec{E}$ and $\vec{H}$ are not in phase in a lossy medium** **(**the conduction current is not negligible**).**

7.2.1. Conducting media (particularly "good conductors")

In conducting media, free electrons will move under the influence of the electric field

$$\vec{j} = \vec{j}_c + \vec{j}_d$$

$$\vec{j}_c = \sigma \cdot \vec{E} \ \text{ and } \vec{j}_d = \frac{\partial \vec{D}}{\partial t};\ \vec{D} = \varepsilon_0 \varepsilon_r \vec{E}$$

A "good conductor" is one for which the conduction current is significantly greater than the displacement current i.e.

$$\left|\vec{j}_c\right| \gg \left|\vec{j}_d\right| = \left|\frac{\partial \vec{D}}{\partial t}\right|$$

➡ allows us to neglect the displacement current term in Maxwell's equations.

$$\boxed{\vec{E} = \vec{E}_0 \cdot e^{i(\omega t - kz)}\ ;\ k = \frac{2\pi}{\lambda} \quad \lambda \text{ - wave length} \Rightarrow \varepsilon \frac{\partial \vec{E}}{\partial t} = \varepsilon \cdot \vec{E}_0 (i\omega) e^{i(\omega t - kz)} = \varepsilon (i\omega) \vec{E} = \vec{j}_d}$$

If all field quantities are sinusoidal of form $e^{i\omega t}$ ➡ $\left|\sigma \cdot \vec{E}\right| \gg \left|\varepsilon \frac{\partial \vec{E}}{\partial t}\right| = \left|i\omega\varepsilon\vec{E}\right|$

Thus for a physical material to be regarded as a good conductor, the frequency satisfies:

$$\sigma \gg \omega\varepsilon \Rightarrow \omega \ll \frac{\sigma}{\varepsilon}$$

➡ This requirement is satisfied for all metals from DC to 100 GHz (and considerably higher in frequency)

7.2.2. Derivation of the differential wave equation in metals

Consider Maxwell's equations in a good conducting medium for which volume charge density is 0 ($\rho \approx 0$) inside the medium and

$$\left|\vec{j}_c\right| \gg \left|\vec{j}_d\right| = \left|\frac{\partial \vec{D}}{\partial t}\right|$$

Maxwell's equations in a conducting medium: $\nabla \vec{D} = \rho \approx 0 \qquad \nabla \vec{B} = 0$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{B} = \mu \vec{j}_c + \varepsilon\mu \frac{\partial \vec{E}}{\partial t} \approx \mu \vec{j}_c \;\; ; \;\; \vec{j}_c = \sigma \cdot \vec{E}$$

From: $\vec{A} \times \left(\vec{B} \times \vec{C}\right) = \vec{B}\left(\vec{A} \cdot \vec{C}\right) - \vec{C}\left(\vec{A} \cdot \vec{B}\right)$

$$\nabla \times (\nabla \times \vec{E}) = \nabla\left(\nabla \vec{E}\right) - \nabla^2 \vec{E} = -\Delta \vec{E}$$

and $\nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\nabla \times \vec{B}\right) = -\frac{\partial}{\partial t}\left(\mu \vec{j}_c\right) = -\mu \cdot \sigma \frac{\partial \vec{E}}{\partial t}$

- one obtain a form of **the wave equation governing propagation in good conductors**:

$$\nabla \times \left(\nabla \times \vec{E}\right) = \nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right) \text{ and } \vec{j}_c = \sigma \cdot \vec{E} \qquad \nabla^2 \vec{E} = \mu\sigma \frac{\partial \vec{E}}{\partial t}$$

7.2.3. Plane wave solution inside the conducting medium

Consider a plane wave propagating in z direction

$$\frac{\partial^2 \vec{E}}{\partial z^2} = \mu\sigma \frac{\partial \vec{E}}{\partial t}$$

- has a solution typical for an attenuated wave, of the form:

$$\vec{E} = \vec{E}_0 e^{i(\omega \cdot t - k \cdot z)} e^{-\alpha \cdot z}$$

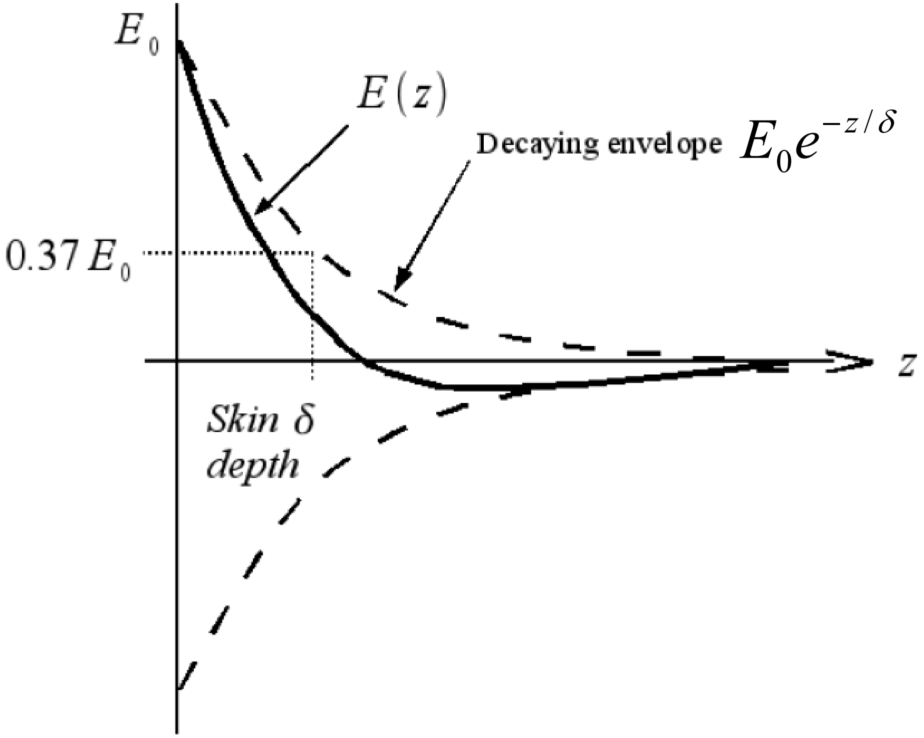
where $e^{-\alpha \cdot z}$

- represents a decaying of the amplitude in the z direction.

By replacing this solution into the wave equation one obtain:

$$\alpha = k = \sqrt{\frac{\omega\mu\sigma}{2}} = \frac{1}{\delta}$$

When z=1/α, the wave amplitude is: $E_0 e^{-1}$



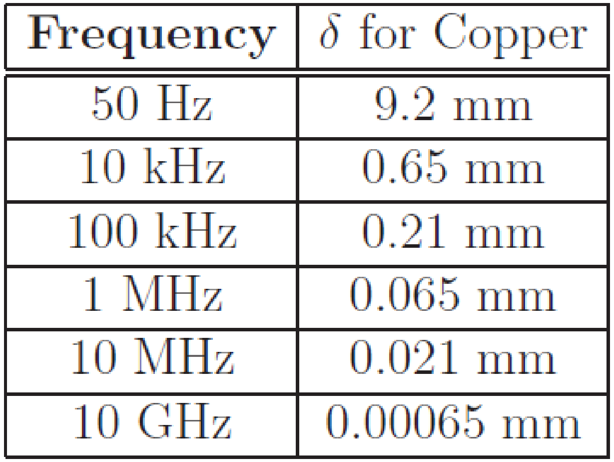
Exponentially decaying plane wave E(z, t = 0) in a “good conductor” medium.

Analysis of the *H* field shows that *H* is 45 degrees out of phase with the *E* field

7.2.4. The skin depth

The exponential decay $e^{-\alpha z}$, may be characterized by the decay constant $1/\alpha$, known as the **“skin depth”**

$$\delta = \alpha^{-1} = \sqrt{\frac{2}{\omega\mu\sigma}} \sim \frac{1}{\sqrt{\omega}}$$



Frequency	δ for Copper
50 Hz	9.2 mm
10 kHz	0.65 mm
100 kHz	0.21 mm
1 MHz	0.065 mm
10 MHz	0.021 mm
10 GHz	0.00065 mm

The skin depth is the distance at which the wave in the medium has decayed to $1/e \approx 0.3678$ of its original amplitude.

At RF and Microwave frequencies, the penetration into metal is small - significant currents flow on **the surface** of good conductors (i.e. concentrated mainly within skin depth) - we talk of the current as being a “**surface current**”.

At high frequencies, the AC current flowing in a conducting wire is concentrated in thin outer layer or “skin” of the wire.
This concentration of current density in a thin outer layer causes the resistance (per metre) to increase as frequency increases.

7.3. DC resistance vs High Frequency (HF) AC resistance

The **DC resistance** of a piece of wire of length l and diameter $2r$ is

$$R_{DC} = \frac{l}{\sigma A} = \frac{l}{\sigma \cdot \pi \cdot r^2} \quad [\Omega] \qquad A = \pi r^2$$ is the cross sectional area.

The **high frequency resistance** of a length of wire of l meters can be crudely estimated by pretending that the current is uniform within an effective conducting area $A_{\text{skin}} = 2\pi r\delta$ (because $\delta \ll r$)

$$R_{HF} \approx \frac{l}{\sigma \cdot A_{skin}} = \frac{l}{\sigma \cdot 2\pi r \cdot \delta} = \frac{l}{\sigma \cdot 2\pi r}\sqrt{\frac{\omega\mu\sigma}{2}} = \frac{l}{2\pi r}\sqrt{\frac{\omega\mu}{2\sigma}} \quad [\Omega] \qquad \text{with:} \quad \delta = \sqrt{\frac{2}{\omega\mu\sigma}}$$

- is not far out in practice for the case where $\delta \ll r$

The high frequency AC resistance can also be expressed as:

$$R_{HF} \approx \frac{l}{\sigma 2\pi r\delta} = R_{DC} \times \frac{r}{2\delta} \quad [\Omega] \quad \Rightarrow \boxed{\frac{R_{HF}}{R_{DC}} = \frac{r}{2\delta}}$$

7.4. Solved and proposed applications

1. Calculate (i) the skin depth, δ, (ii) the **resistance per meter** at DC and at 50 MHz of a copper wire of diameter 5 mm and (iii) calculate the ratio between R_{HF}/R_{DC} .

   $r = 2.5 \times 10^{-3}$ m and $\rho=1.72\cdot10^{-8}\ \Omega$m.

We may consider the equation R_{HF}/R_{DC} derived in Section 7.3.

We find in DC: $\frac{R_{DC}}{l} \approx 0.0008\ \Omega\text{m}^{-1}$

At 50 MHz: $\frac{R_{HF}}{l} \approx \left(\frac{R_{DC}}{l}\right)\frac{r}{2\delta} \approx 8\times10^{-4}\,\frac{2.5\times10^{-3}}{2.5\times10^{-5}} = 0.1\ \Omega\text{m}^{-1}$ ➡ R_{HF}/R_{DC} =…

R_{HF} is 125 times greater than at DC

Question: how can we surpass this problem?

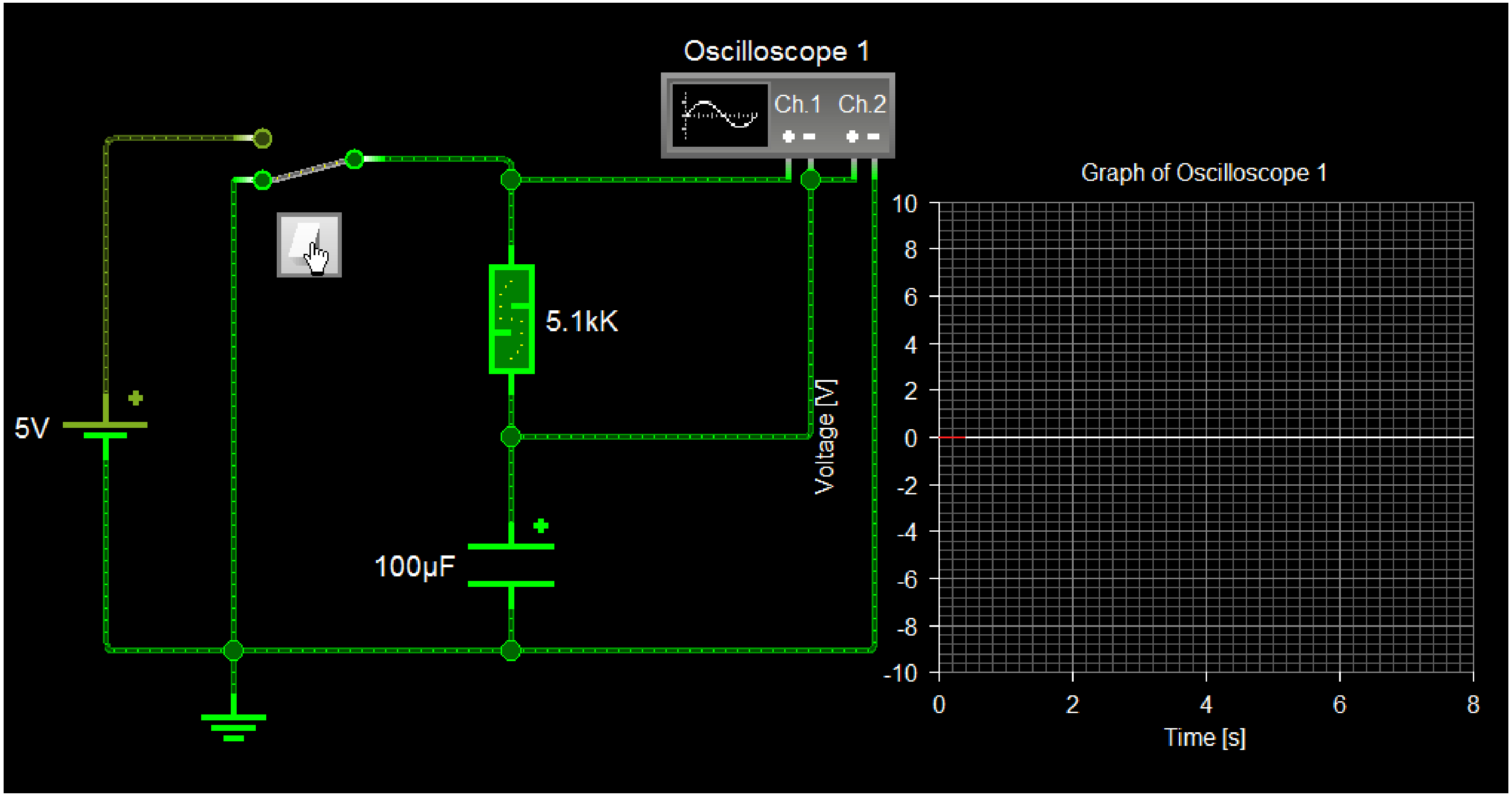
A type of cable called litz wire (from the German *Litzendraht*, braided wire) is used to mitigate the skin effect for frequencies of a few kilohertz to about one megahertz. It consists of a number of insulated wire strands woven together in a carefully designed pattern, so that the overall magnetic field acts equally on all the wires and causes the total current to be distributed equally among them. With the skin effect having little effect on each of the thin strands, the bundle does not suffer the same increase in AC resistance that a solid conductor of the same cross-sectional area would due to the skin effect.

Litz wire is often used in the windings of high-frequency transformers to increase their efficiency

2. A plane electromagnetic wave with frequency $\nu=10^6$ Hz falls onto a conductive body with $\varepsilon_r=1$, $\mu_r=1$, and conductivity $\sigma=100\ \Omega^{-1}\text{m}^{-1}$. Compute the skin depth δ of the wave in the conductive body. Take $\mu_0=4\pi\cdot10^{-7}$ H/m.

3. An AC electric field with frequency f=50 Hz generates a current through a conductor of resistivity $\rho=1.78\cdot10^{-8}\ \Omega\cdot\text{m}$. Calculate the ratio between the displacement current and the conduction current. We assume that $\varepsilon=\varepsilon_0=8.854\times10^{-12}$ C/V m and the current is uniform distributed inside the conductor. What is this ratio if f=10 kHz and 100 kHz?

Chapter VIII. Passive electrical components under time varying currents



https://circuitdigest.com/tutorial/rc-rl-and-rlc-circuits

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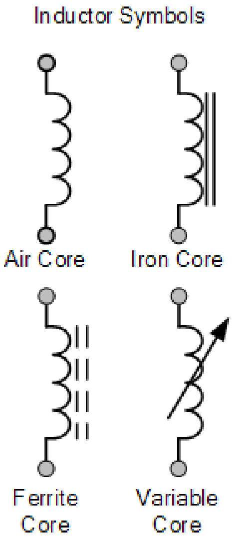
8.1. Inductors

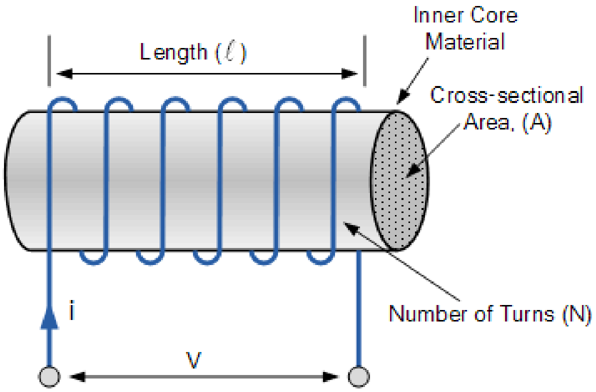
8.1.1. Self inductance

An **Inductor** is a coil of wire wound around a central core. For most coils the current, *i*, flowing through the coil produces a magnetic flux, ($N\Phi$) around it that is proportional to this flow of electrical current.

Inductors are formed with wire tightly wrapped around a solid central core which can be either a straight cylindrical rod or a continuous loop or ring to concentrate their magnetic flux.

Inductor Symbol:





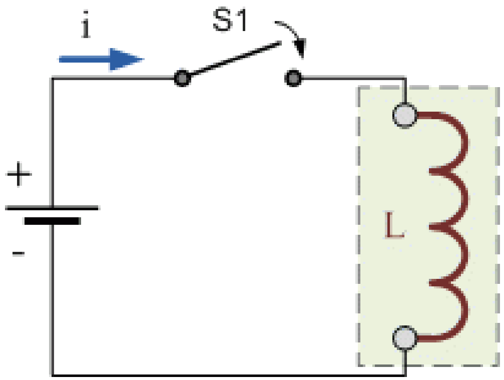
A typical inductor

The current, *i*, that flows through an inductor produces a magnetic flux that is proportional to it.

We suppose that i(t).

Because of the e.m. induction law, an inductor opposes the rate of change of current flowing through it due to the self-induced e.m.f.

Inductors resist or oppose changes of current but will easily pass a steady state DC current.



Consider an isolated coil, loop, or circuit with <u>fixed geometry</u> in which a current i(t) flows.
<u>This current produces a magnetic field that links the circuit itself</u>.
If the current *i* varies with time, so <u>does the flux linkage</u>
➔ results a <u>self-induced e.m.f in the circuit</u>

<u>Sometimes this e.m.f. is referred as a back-e.m.f because its direction is that opposes any change in the driving emf (Lentz's law).</u>

$$U_L = -\frac{d\left(N\varphi_1\right)}{dt} \quad \text{with} \quad \Phi = N\varphi_1 = i \cdot L$$

⇓

$$U_L = -L\frac{di}{dt}; \quad L = \frac{\Phi}{i} = \frac{N\varphi_1}{i} \quad \text{[H] "Henry"}$$

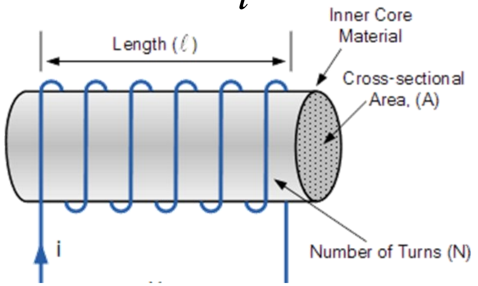
N – number of turns (surfaces)
φ_1 – the magnetic flux through <u>1 surface</u>
L – <u>self inductance (inductance)</u>
L is constant if the magnetic properties of the core (μ) are constant (note that for ferromagnetic materials, μ=f(H))

<u>Example</u> - <u>the self inductance of a solenoid</u>: $L = \frac{\Phi}{i}; \quad \Phi = N \cdot B \cdot A = N\frac{\mu N i}{l}A$

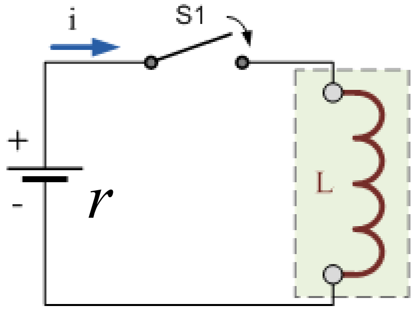
➔ $$\boxed{\begin{aligned} L &= \frac{\mu N^2 A}{l} \\ A &= \pi R^2 \end{aligned}} \qquad \boxed{L = \frac{\mu N^2 A}{l} \cdot f}$$

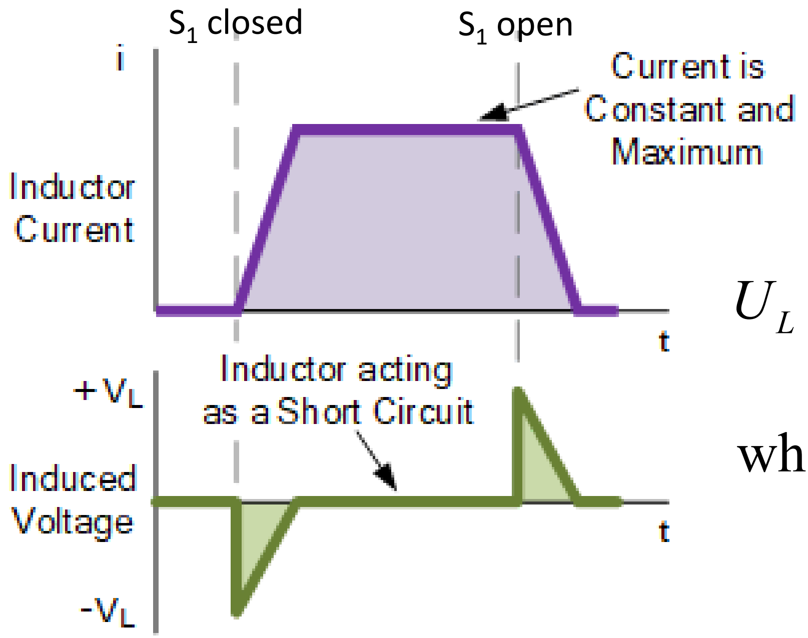
A – cross section area

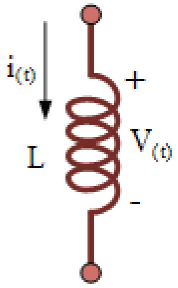
f=0.53 for *l*/*R*=1
f=0.85 for *l*/*R*=5
geometrical factor



Current and Voltage in an Inductor







$$U_L = -L\frac{di}{dt}$$

$$\text{when } \frac{di}{dt} = 0 \Rightarrow U_L = 0$$

With a steady state DC current flowing through the inductor and therefore zero induced voltage across it, the inductor acts as a short circuit equal to a piece of wire, or at the very least a very low value resistance. In other words, the opposition to the flow of current offered by an inductor is very different between AC and DC circuits.

The polarity of the induced voltage is given by the Lenz's law.

8.1.2. Magnetic field energy

Consider an inductor, *L*, connected to an external source of e.m.f.
An inductor in a circuit opposes the flow of current, *i*, through it because the flow of this current induces an emf that opposes it (Lenz's Law). Work has to be done by the external source in order to keep the current flowing against this induced emf. The instantaneous power used in forcing the current, *i*, against this self-induced emf, U_L, is :

$$P = u \cdot i = \left(L \frac{di}{dt} \right) i = L \cdot i \frac{di}{dt} = \frac{d}{dt} \left(\frac{1}{2} L i^2 \right) \quad [W]$$

An ideal inductor has only inductance and no resistance (R = 0 Ω) and therefore no power is dissipated within the coil.
We can say that an ideal inductor has zero power loss.

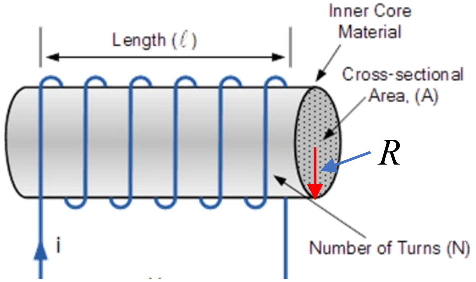
We suppose that the resulting current i(t) increases in some unspecified manner from 0 to a final value I.
The total amount of (electrical) work done by the source is:

$$W_S = \int P dt = L \int_0^I i di = \frac{1}{2} L I^2 \quad [J] \qquad \Rightarrow \qquad W_M = \frac{1}{2} L I^2 \quad [J]$$

- the work done does not depend on the way in which i(t) varies to achieve its final value I
- when the steady-state condition is reached, we have di/dt=0 and the power is no longer expended by the source.
- when current flows into an inductor, energy is stored in its magnetic field: W_M .

Example of calculation of the magnetic energy for a solenoid:

The following relations were found for a solenoid:



$$L = \frac{\mu N^2 A}{l}\,;\;\; B = \frac{\mu N I}{l} \Rightarrow I = \frac{l \cdot B}{\mu N}$$

$A = \pi R^2$ $\quad R$ - radius

⬇

$$W_M = \frac{1}{2} L I^2 \;\; [J] \quad ➡ \quad W_M = \frac{B^2}{2\mu}\pi R^2 l = w_M V \;\; \Rightarrow \boxed{w_M = \frac{B^2}{2\mu} = \frac{1}{2}\vec{H}\vec{B} \quad [J/m^3]}$$

w_M is the energy density of the magnetic field - this is a local term i.e., the formula has a very general character.

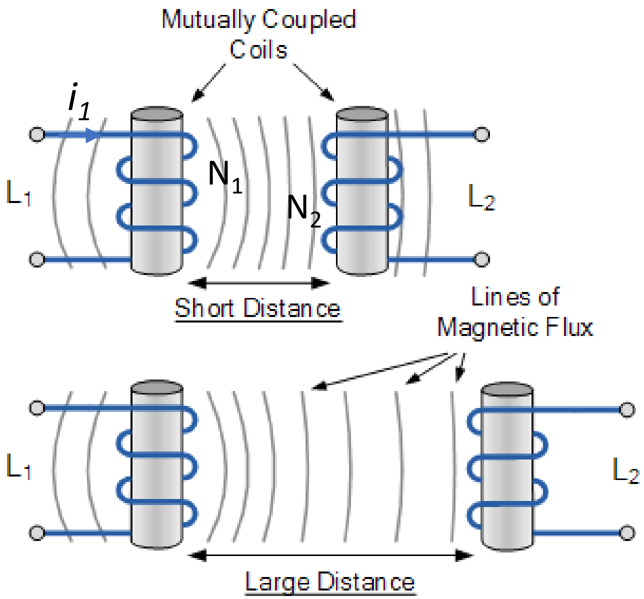
Now, we can remember the expression for the energy density of the electric field:

$w_E = \frac{1}{2}\vec{E}\cdot\vec{D}$ $\quad E$ – electric field strength; D – electric displacement vector; $\vec{D} = \varepsilon_0 \varepsilon_r \vec{E}$

➡ $\boxed{w_{EM} = \frac{1}{2}\vec{E}\vec{D} + \frac{1}{2}\vec{H}\vec{B}}$ $\quad w_{EM}$ is the energy density of the electromagnetic field

8.1.3. Mutual inductance

Mutual Inductance is the interaction of one coil's magnetic field on another coil as it induces a voltage in the adjacent coil



The changing current in the primary coil (1) induces an e.m.f. in the secondary coil (2).

$$\varphi_{21} = \iint\limits_{S_2} \vec{B}_1 dA_2 \,; \quad \Phi_{21} = N_2 \varphi_{21}$$

$$\Phi_{21} = M_{21} \cdot i_1$$

- the magnetic flux created by magnetic circuit (1) in coil (2)

M_{21} is a strictly geometric factor – **mutual inductance**

$$\boxed{U_{21} = -M_{21} \frac{di_1}{dt}}$$

The amount of mutual inductance that links one coil to another depends very much on the relative positioning of the two coils.

The mutual inductance that exists between the two coils can be greatly increased by positioning them on a common soft iron core or by increasing the number of turns of either coil as would be found in a **transformer**.

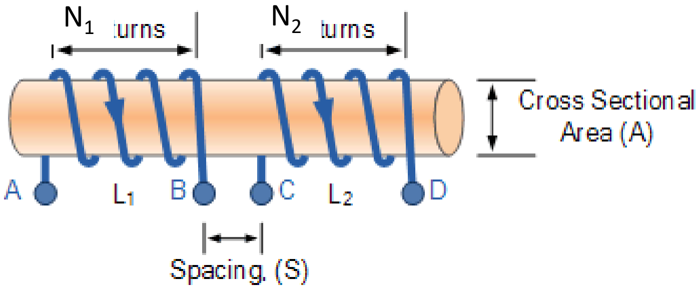
We can reverse the situation: the coil 2 will produce a magnetic field, B_2, and hence a magnetic flux φ_{12} through the coil 1.
Such that:

$$U_{12} = -M_{12}\frac{di_2}{dt} \quad with \quad M_{12} = \frac{N_1\varphi_{12}}{i_2}$$

It can be shown that $M_{12}=M_{21}=M$ - which is named **mutual inductance**

1 H=1V/(A·s^{-1})=1 Wb/A (1 Wb=1 T · m^2)

Assume two coils that are tightly wound one on top of the other and/or over a common soft iron core. Assuming a perfect flux linkage between the two coils the mutual inductance that exists between them can be given as:



$$\Phi_{21} = N_2 B_1 \cdot A = N_2 \frac{\mu N_1 i_1}{l} A \Rightarrow M_{21} = \frac{\Phi_{21}}{i_1} = \frac{\mu N_1 N_2 A}{l}$$

$$\Phi_{12} = N_1 B_2 \cdot A = N_1 \frac{\mu N_2 i_2}{l} A \Rightarrow M_{12} = \frac{\Phi_{12}}{i_2} = \frac{\mu N_1 N_2 A}{l}$$

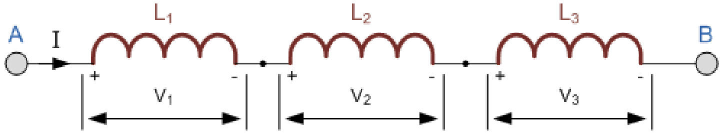
$$\Rightarrow M_{12} = M_{21} = M$$

$$\boxed{L_1 = \frac{\mu N_1^2 A}{l}\ ;\ L_2 = \frac{\mu N_2^2 A}{l} \Rightarrow M = \sqrt{L_1 L_2}}$$

- this equation assumes zero flux leakage and 100% magnetic coupling between the two coils, L_1 and L_2.

In practice $M = k\sqrt{L_1 L_2}$ k - coefficient of coupling; $0 \le k \le 1$

8.1.4. Inductors connected in series



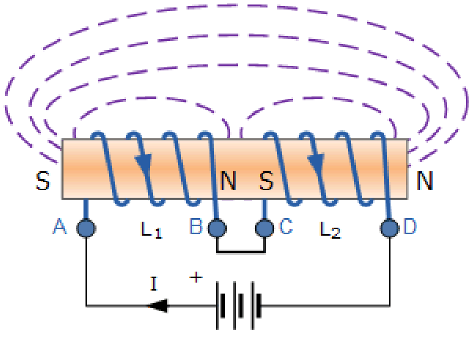
$$U_S = U_1 + U_2 + \dots$$

$$L_S \frac{di}{dt} = L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + \dots. \qquad (1)$$

$$\Rightarrow \quad \boxed{L_S = L_1 + L_2 + L_3 + \dots.} \qquad (2)$$

Mutually Connected Inductors in Series

Cumulatively Coupled Series Inductors



Eq. 2 needs to take into account the interaction between the two coils due to the effect of mutual inductance.
The self inductance of each individual coil, L1 and L2 respectively will be the same as before but with the addition of M denoting the mutual inductance.

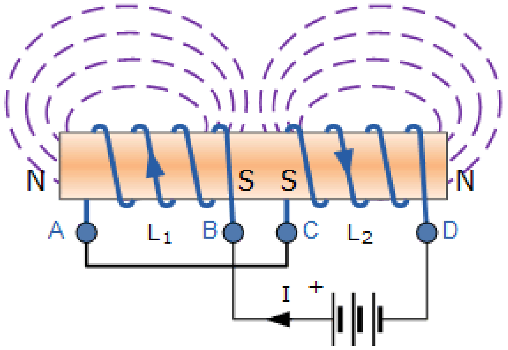
Eq. 1 becomes:

$U_S = U_1 + U_2 + U_{coupl}$ - the total emf induced into the cumulatively coupled coils

$$L_S \frac{di}{dt} = L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + 2\left(M \frac{di}{dt} \right) \Rightarrow L_S = L_1 + L_2 + 2M$$

$2M$ represents the influence of coil L_1 on L_2 and likewise coil L_2 on L_1.

Differentially Coupled Series Inductors

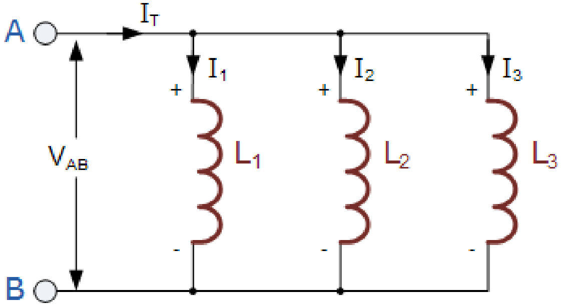


$$U_S = U_1 + U_2 - U_{coupl}$$

$$L_S \frac{di}{dt} = L_1 \frac{di}{dt} + L_2 \frac{di}{dt} - 2\left(M \frac{di}{dt} \right) \Rightarrow L_S = L_1 + L_2 - 2M$$

- the general equation for inductively coupled inductors in series is: $\boxed{L_S = L_1 + L_2 \pm 2M}$

8.1.5. Inductors in Parallel



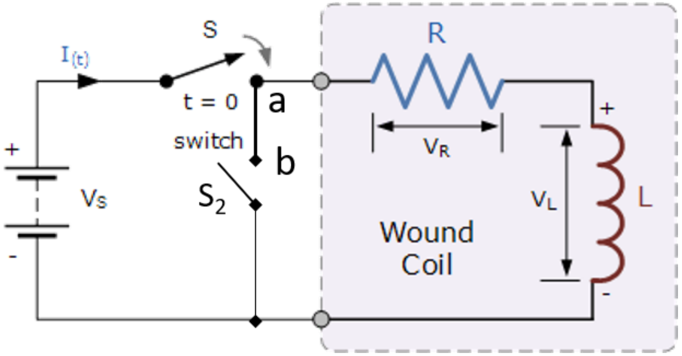
$$U_{AB} = L_P \frac{di}{dt} = L_1 \frac{di_1}{dt} = L_2 \frac{di_2}{dt} = \ldots.$$

$$\frac{di}{dt} = \frac{U_{AB}}{L_P}\,;\ \frac{di_1}{dt} = \frac{U_{AB}}{L_1}\,;\ \frac{di_2}{dt} = \frac{U_{AB}}{L_2}\ \ldots$$

$$\boxed{\frac{1}{L_p} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \ldots}$$

From Kirchhoff's first law: $i = i_1 + i_2 + i_3 + \ldots$

8.2. RL series circuit

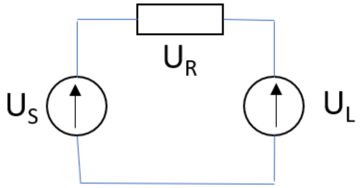


A **RL Series Circuit** consists basically of an inductor of inductance, L connected in series with a resistor of resistance, R. The resistance “R” is the DC resistive value of the wire

a) Energizing an inductor

At t=0 the switch S is closed: $U_S = U_R + U_L \Rightarrow U_S = L\frac{di}{dt} + iR$

Because of the changing current in the inductor, there is produced a back e.m.f., L·di/dt, that opposes driving e.m.f. U_S.

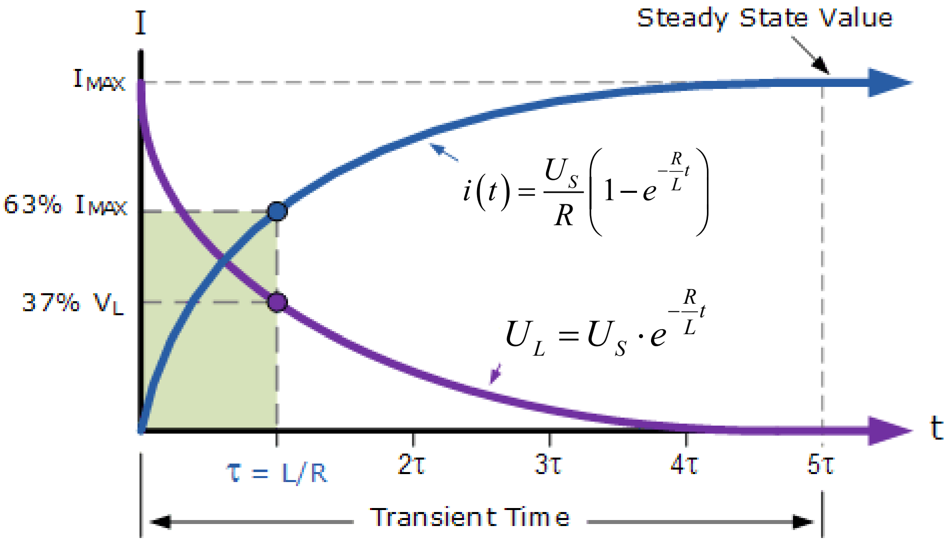


$$U_S = L\frac{di}{dt} + iR \Rightarrow L\frac{di}{dt} = U_S - iR \Rightarrow L\cdot di = \left(U_S - iR\right)dt \Rightarrow\ L\frac{di}{U_S - iR} = dt$$

$$L\int\frac{di}{U_S - iR} = \int dt\ \Rightarrow\ -\frac{L}{R}\ln\left(U_S - iR\right) = t + C \Rightarrow \ln\left(U_S - iR\right) = -t\frac{R}{L} + C_1 \Rightarrow U_S - iR = e^{-t\frac{R}{L}}\cdot C_2$$

For t=0, i=0 ➡ $U_S = C_2$ ➡ $\boxed{i(t) = \frac{U_S}{R}\left(1 - e^{-t\frac{R}{L}}\right)}$ $\tau = \frac{L}{R}$ [s] time constant

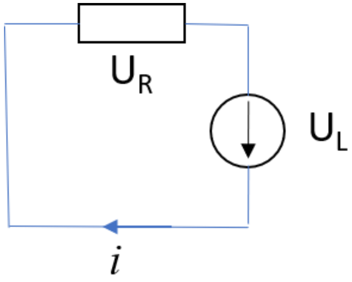
The transient curves when the LR series circuit is energized:



$$U_L=-L\frac{di}{dt}\Rightarrow U_L=U_S\cdot e^{-\frac{R}{L}t}$$

b) De-energizing an inductor

After the inductor has been fully energized, the switch S_2 is moved to position b and S is open.



$$L\frac{di}{dt}+iR=0\Rightarrow L\frac{di}{dt}=-iR\Rightarrow\frac{di}{i}=-\frac{R}{L}dt$$

- after integrating this differential equation:

$$\ln i=-\frac{R}{L}t+C_1\Rightarrow i(t)=e^{-\frac{R}{L}t}\cdot C_2$$

Now, for t=0 ➔ $i(0)=U_S/R$ ➔ $C_2=U_S/R$ ➡ $\boxed{i(t)=\frac{U_S}{R}e^{-\frac{R}{L}t}}$ Can you plot i(t)?

c) Energy balance

The process of energizing the inductor

From:

$$U_S = L\frac{di}{dt} + iR \quad \Big|\cdot idt \quad \Rightarrow \quad \underbrace{d\left(\frac{1}{2}Li^2\right)}_{\text{The energy stored in magnetic field}} + \underbrace{i^2Rdt}_{\text{Joule heating}} = \overbrace{U_S idt}^{\text{The work done by the source}}$$

The process of de-energizing the inductor

In a similar way one obtain:

$$-d\left(\frac{1}{2}Li^2\right) = i^2Rdt$$

- the removed magnetic energy appears in the form of Joule heat dissipated by the resistor

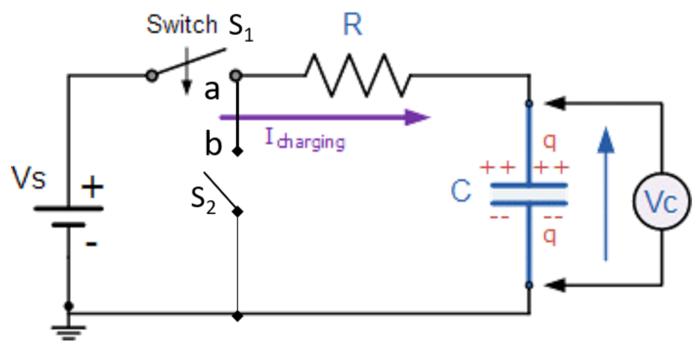
d. Simple proposed application

A coil which has an inductance of 40 mH and a resistance of 2 Ω is connected together to form a LR series circuit like presented in Section 8.2. If they are connected to a 20 V DC supply, calculate:

a) the final steady state value of the current; b) the time constant of the RL series circuit; c) the transient time of the RL series circuit; d) the value of the induced emf in inductor after 10 ms; e) the value of the circuit current one time constant after the switch is closed.

8.3. RC – series circuits

a) RC Charging Circuit



When a voltage source is applied to an RC circuit, the capacitor, C, charges up through the resistance, R.

Suppose that:

$t = 0 \Rightarrow q = 0$ $\quad q$ – charges on the plates

Now, at t=0 S_1 is closed ➔ i(t)=?

From: $i = \dfrac{dq}{dt}$

$U_S = U_R + U_C$ ➡ $U_S = iR + \dfrac{q}{C} \Rightarrow U_S = R\dfrac{dq}{dt} + \dfrac{q}{C} \Rightarrow U_S - \dfrac{q}{C} = R\dfrac{dq}{dt}$

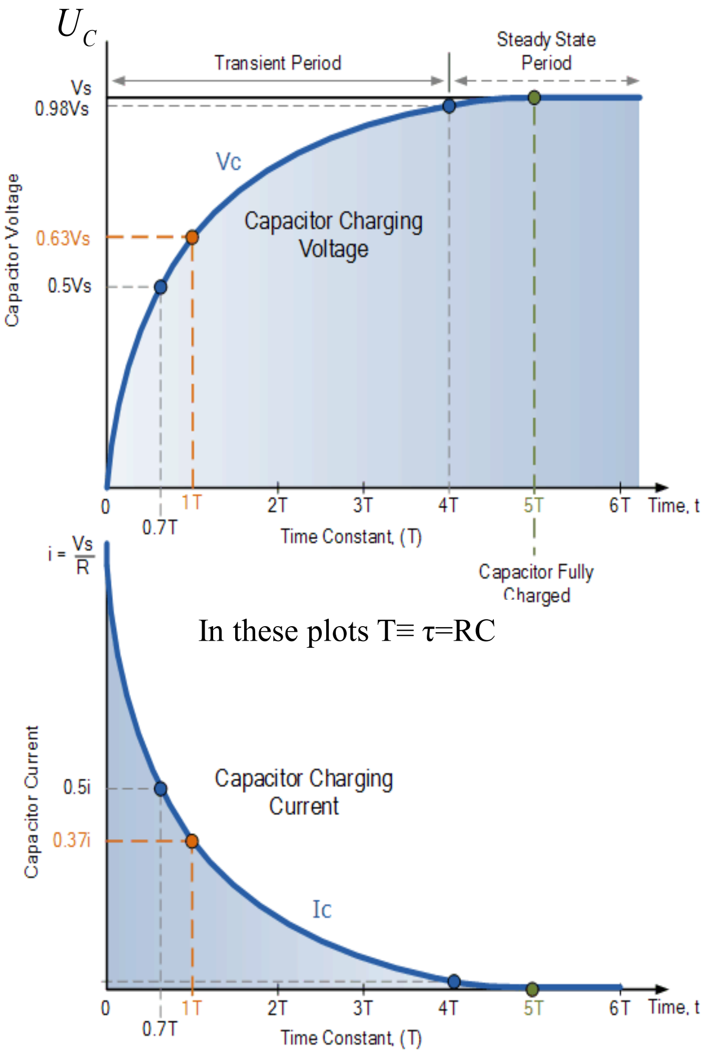
we find:

$C \cdot U_S - q = RC\dfrac{dq}{dt} \Rightarrow \displaystyle\int \frac{dq}{q - C \cdot U_S} = -\int \frac{1}{RC}dt$ ➡ $\ln\left(q - C \cdot U_S\right) = -\dfrac{t}{RC} + K, \quad K = const.$

➡ $q - C \cdot U_S = e^{-\frac{t}{RC}} e^{K} = e^{-\frac{t}{RC}} K_1$ - from initial condition: $t = 0 \Rightarrow q = 0 \Rightarrow -C \cdot U_S = K_1$

➡ $q = C \cdot U_S - C \cdot U_S\, e^{-\frac{t}{RC}} \Rightarrow q = C \cdot U_S\left(1 - e^{-\frac{t}{RC}}\right)$ and $\boxed{U_C = \dfrac{q}{C} = U_S\left(1 - e^{-\frac{t}{RC}}\right)}$

The time behaviour of U_C and i:



In these plots T≡ τ=RC

$$U_C = \frac{q}{C} = U_S\left(1 - e^{-\frac{t}{RC}}\right)$$

$\tau = RC$ - time constant

$$i = \frac{dq}{dt} = \frac{U_S}{R} e^{-\frac{t}{RC}}$$

$$t = 0 \Rightarrow i = \frac{U_S}{R}$$

$$t \to \infty \Rightarrow i \to 0$$

For R=1 kΩ and C=100 nF ➔ τ=100 μs

$l = c \cdot \tau$ =300 m

The characteristic length $l=c\cdot\tau$

A circuit with ordinary size should be much smaller than c·τ in order to neglect the radiation and phase-shift difference effects; c is speed of light.

b. Discharging the capacitor

- at t=0 S_1 is open and S_2 is closed (in poz. b, <u>see the circuit</u>) ➔ i(t)=?

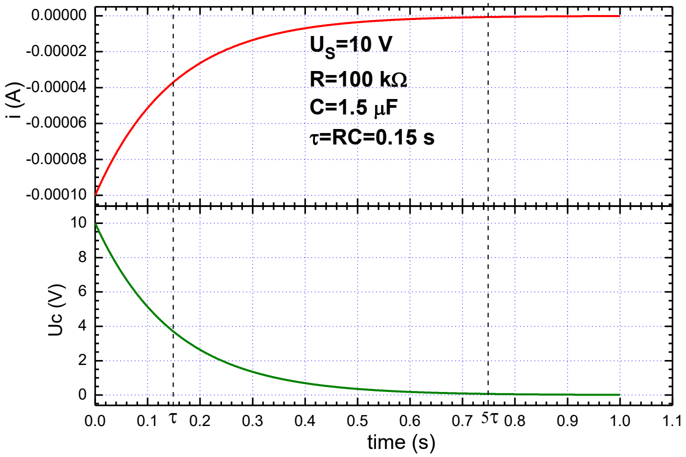
$U_S=0$ ➔ $0 = iR + \frac{q}{C} \Rightarrow 0 = R\frac{dq}{dt} + \frac{q}{C} \Rightarrow \frac{dq}{dt} = -\frac{q}{RC}$ ➔ $\frac{dq}{q} = -\frac{1}{RC}dt$

$$i = \frac{dq}{dt}$$

By integrating the differential equation $\int\frac{dq}{q} = -\frac{1}{RC}\int dt$

we find: $\ln q = -\frac{t}{RC} + K\,,\; K = const. \Rightarrow q = e^{-\frac{t}{RC}}e^{K} \Rightarrow q = e^{-\frac{t}{RC}}\cdot K_2$

$$t = 0 \Rightarrow q(0) = q_0 = C\cdot U_S \Rightarrow K_2 = C\cdot U_S$$



The time behaviour of i and U_C

$q(t) = C\cdot U_S\cdot e^{-\frac{t}{RC}}$ and $i = \frac{dq}{dt} \Rightarrow i(t) = -\frac{q_0}{RC}\cdot e^{-\frac{t}{RC}}$

if $q_0 = C\cdot U_S$

$i(t) = -\frac{C\cdot U_S}{RC}\cdot e^{-\frac{t}{RC}} \Rightarrow i(t) = -\frac{U_S}{R}\cdot e^{-\frac{t}{RC}}$

- <u>the discharge current changes the sign!</u>

$U_C = \frac{q}{C} = U_S\cdot e^{-\frac{t}{RC}}$ - the voltage over the capacitor

c. Energy balance

By following the same approach like for RL circuits, we find:

Charging:

$$\frac{q}{C}+iR=U_S \;\Big|\cdot\frac{dq}{dt} \;\Rightarrow\; \frac{q}{C}\cdot\frac{dq}{dt}+i^2R=U_S\cdot i$$

$$d\left(\frac{1}{2}\frac{q^2}{C}\right)+i^2Rdt=U_S\cdot i\cdot dt$$

- the energy delivered by the source

- the Joule heating in resistor

- the energy stored by the capacitor

Discharging:

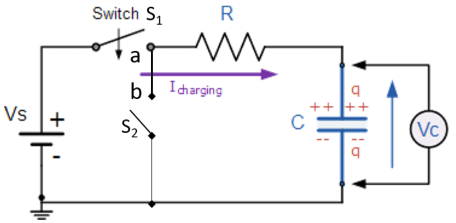
$$\frac{q}{C}+iR=0 \quad \Big|\cdot\frac{dq}{dt}=i$$

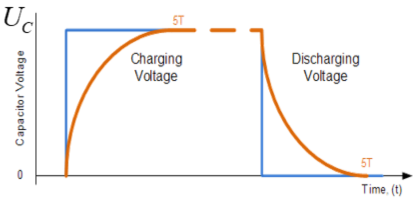
$$-d\left(\frac{1}{2}\frac{q^2}{C}\right)=i^2Rdt$$

- the energy delivered by the capacitor is found in Joule heating of the resistor

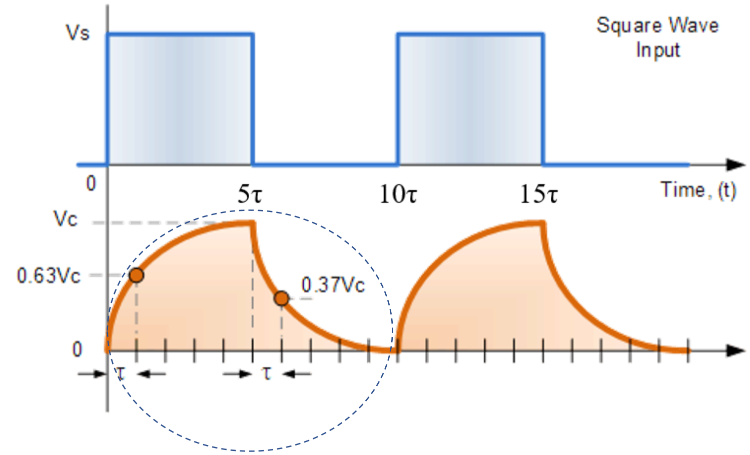
8.3.1. Basic applications of RC circuits

Typical RC Waveforms from the circuit bellow





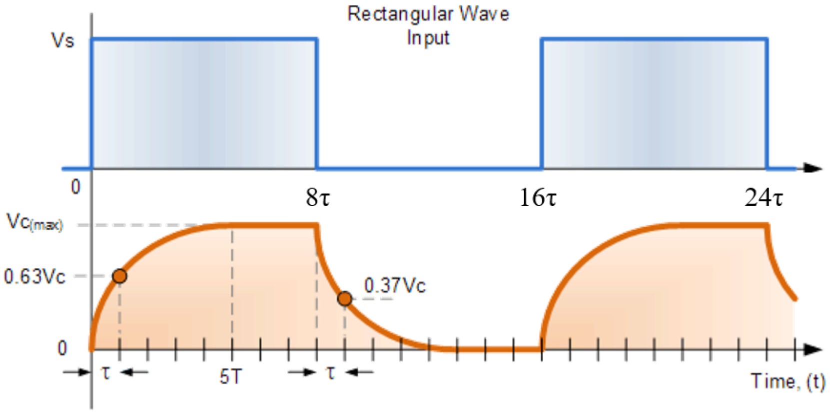
A 5RC period Input Waveform:



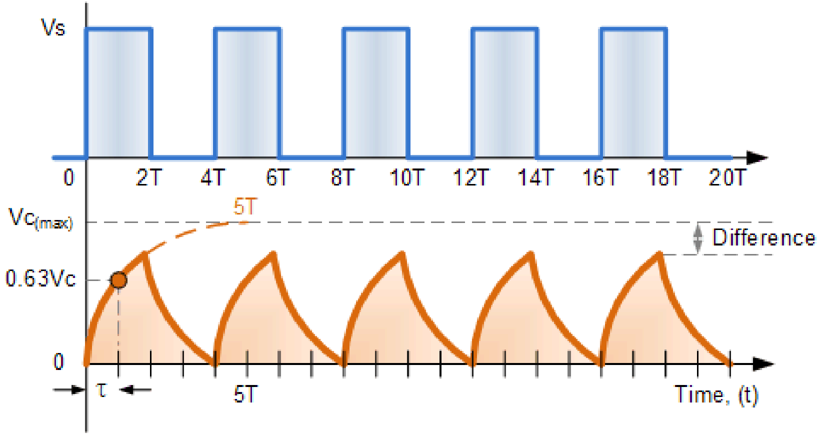
In this this example, the frequency (and therefore the resulting time period, $f = 1/T$) of the input square wave voltage waveform exactly matches twice that of the 5RC time constant

➔T=10τ

A Longer, 8RC Input Waveform

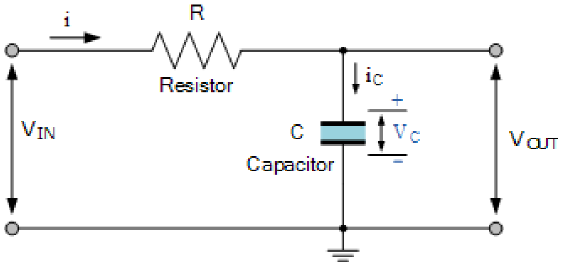


A Shorter, 4RC Input Waveform



a) RC Integrator

The RC integrator is a series connected RC network that produces an output signal which corresponds to the mathematical process of integration.



$$U_C = \frac{q}{C} = U_{in}\left(1 - e^{-\frac{t}{RC}}\right)$$

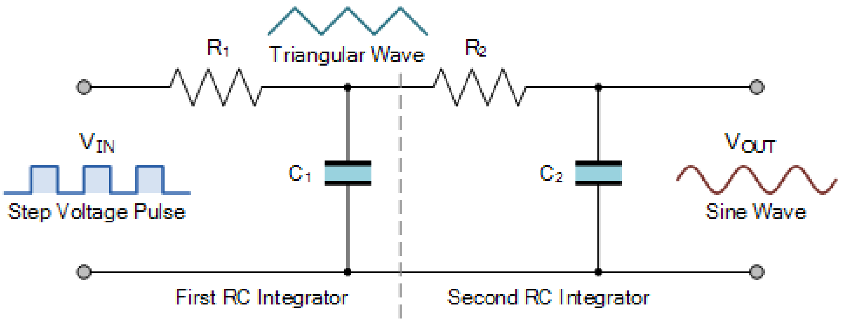
$\tau = RC$ - time constant

$$\mathrm{U}_{out} = \mathrm{U}_C = \frac{\int i dt}{C}, \quad \text{with } i = \frac{U_{in}}{R} e^{-\frac{t}{RC}} \Rightarrow \mathrm{U}_{out} = \frac{1}{RC}\int U_{in} e^{-\frac{t}{RC}} dt$$

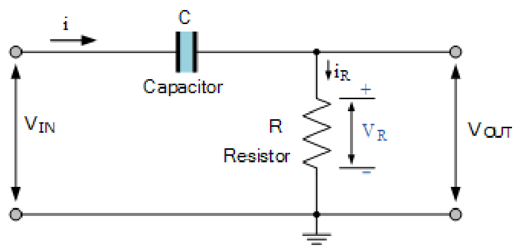
⇓ by proper adjustment of RC components

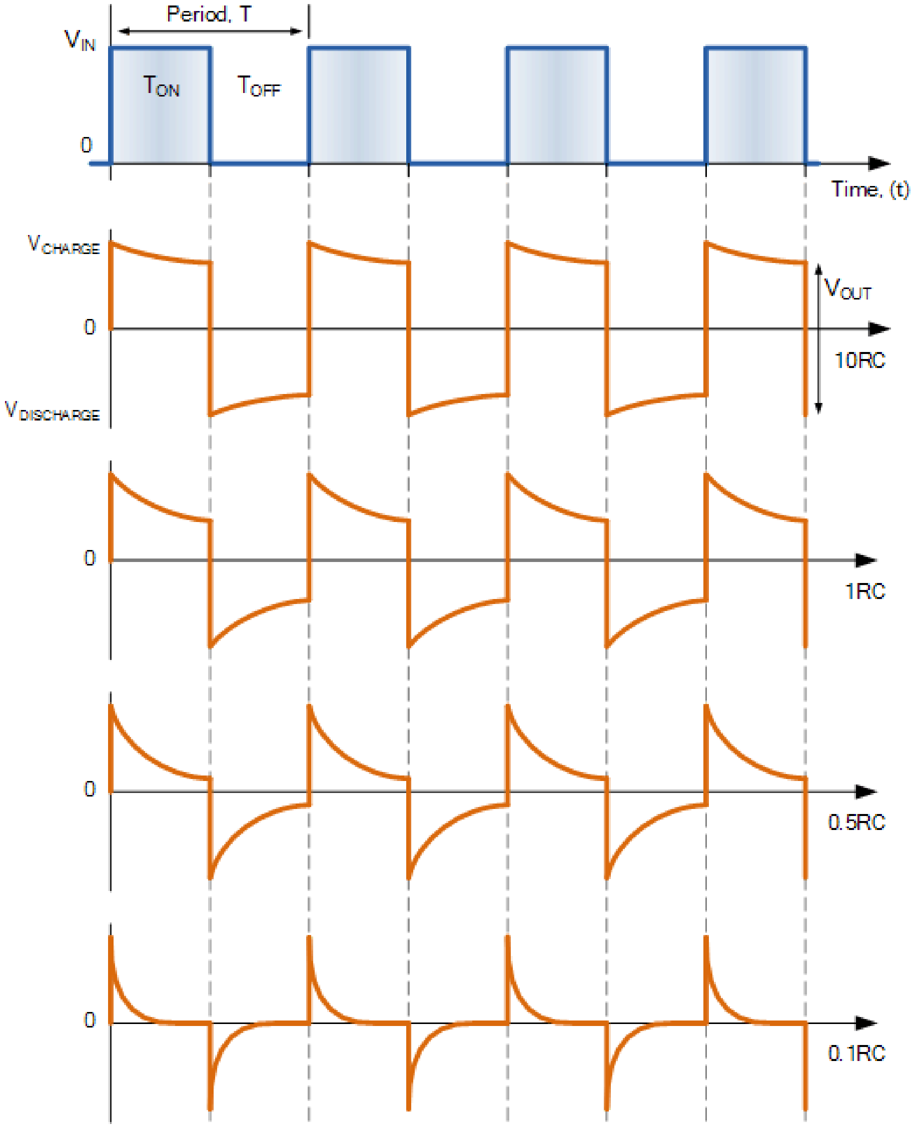
$$\mathrm{U}_{out} \simeq \frac{1}{RC}\int U_{in} dt$$

Sine Wave RC Integrator



b. The RC differentiating circuit





From:

$$\frac{q}{C}+iR=U_{in} \quad \Rightarrow q+iRC=C\cdot U_{in}$$

By design, *RC can be sufficiently small* such that *iRC* can be neglected

$$q \approx C\cdot U_{in}$$

$$U_{out}=i\cdot R=R\frac{dq}{dt}\approx RC\frac{dU_{in}}{dt}$$

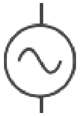
$$U_{out}=0 \;\text{ for }\; U_{in}=ct.$$

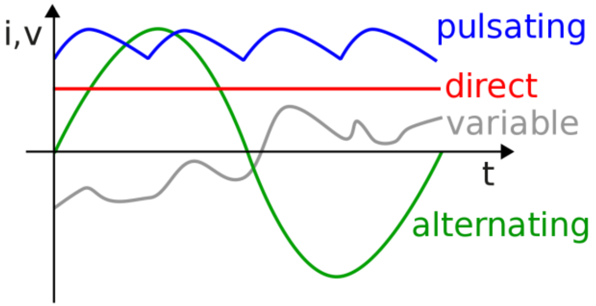
If the input were a perfect square wave and if the circuit behaves like an ideal one, RC→0, the output, U_{out}, would consist of sharp spikes with essentially zero width.

8.4. RLC series circuits in sine Alternating Currents (AC)

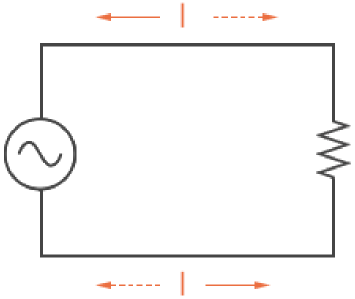
8.4.1. Introduction

In what follows, we shall consider sine AC voltages/currents applied to passive electrical components like R, L, C or different combinations of such elements

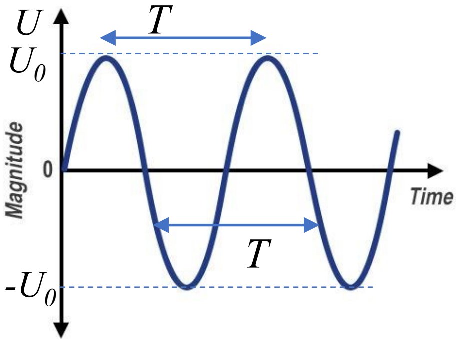
An AC voltage is produced by a power generator, like we described in Section 4.6.2, or by an electronic oscillator.
The symbol of such a source which can be applied to a load is: 



Different types of voltage/current signals



Alternating Current (AC)



The AC voltage or current can be expressed by:

$$\begin{cases} U(t) = U_0 \sin(\omega \cdot t + \varphi_0)\ ;\ \omega = \dfrac{2\pi}{T} \\ I(t) = I_0 \sin(\omega \cdot t + \varphi_0) \end{cases}$$

with: T – signal period; ω – pulsation; φ_0 - initial phase; U_0, I_0 – the amplitude; t - time

8.4.2. Purely resistive load in AC

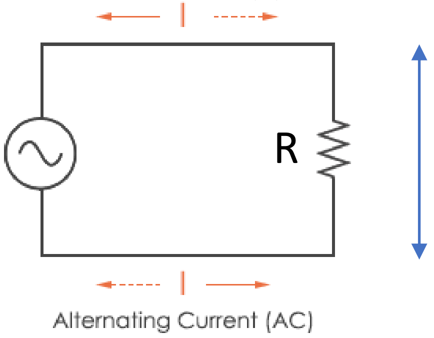
Consider a resistive load connected to an AC generator $U(t) = U_0 \sin(\omega \cdot t)$

The current that flows through the resistor, R, is: $I_R(t) = I_{R0} \sin(\omega \cdot t + \varphi)$

where φ is a possible phase difference between voltage and current

$$I_R(t) = I_{R0} \sin(\omega \cdot t + \varphi)$$

$$U(t) = U_0 \sin(\omega \cdot t)$$



$$U_R(t)$$

$$\Rightarrow U(t) - I_R(t) R = 0$$

$$U_R(t) = I_R(t) R$$

- instantaneous voltage across the R

$$I_R(t) = \frac{U(t)}{R} = \frac{U_0 \sin \omega t}{R} = I_{R0} \sin \omega t \quad \text{with} \quad I_{R0} = \frac{U_{R0}}{R} = \frac{U_{R0}}{X_R}$$

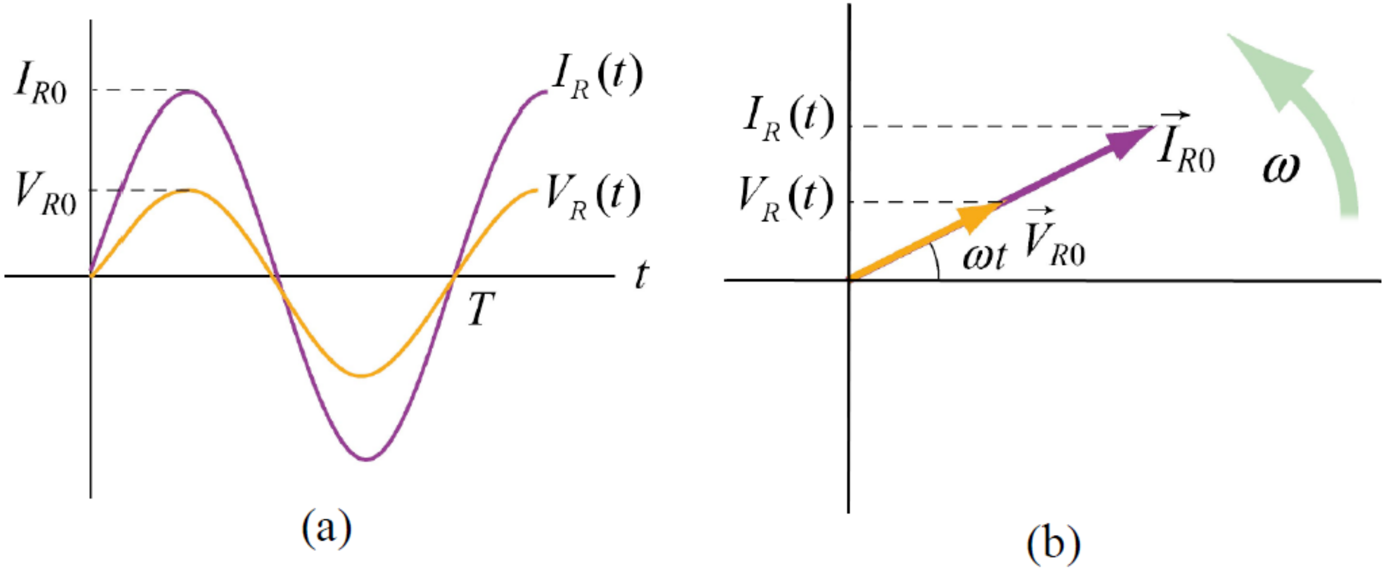
<u>the amplitude of the current</u>

X_R – <u>resistive reactance</u>

<u>Observations</u>:

- the amplitude of the current is <u>independent of the pulsation (or frequency)</u>
- φ=0 ➔ I_R and U are in phase.

This behaviour is plotted bellow:



a) the time dependence of U_R and I_R and b) the Phasor diagram of U_R and I_R .

Phasor is a rotating vector characterized by:

- length - represents the amplitude;
- angular speed - the vector rotates counter clockwise with ω;
- the projection of the vector over y axis (vertical axis) represents his instantaneous value at time t: $U_R(t) = U_{R0} sin\omega t$; the same can be expressed for $I_R(t) = I_{R0} sin\omega t$;
- the phasor is represented using the vector sign, e.g., $\vec{U}_{R0}$ or $\vec{I}_{R0}$

The average value of the voltage and current over one period can be expressed as:

$$\langle U_R\rangle = \frac{1}{T}\int_0^T U_R(t)\,dt = \frac{1}{T}U_{R0}\int_0^T \sin(\omega t)\,dt = 0 \qquad \text{because} \qquad \frac{1}{T}\int_0^T \sin(\omega t)\,dt = 0$$

The same is true for current:

$$\langle I_R\rangle = \frac{1}{T}\int_0^T I_R(t)\,dt = \frac{1}{T}I_{R0}\int_0^T \sin(\omega t)\,dt = 0$$

However, *the time average of the square* of the voltage and current can be expressed by:

$$\left\langle U_R^2\right\rangle = \frac{1}{T}\int_0^T U_R^2(t)\,dt = \frac{1}{T}U_{R0}^2\int_0^T \sin^2(\omega t)\,dt = \frac{1}{2}U_{R0}^2\,;\ \left\langle I_R^2\right\rangle = \frac{1}{2}I_{R0}^2 \qquad \text{because} \quad \frac{1}{T}\int_0^T \sin^2(\omega t)\,dt = \frac{1}{2}$$

So, we can define the *root-mean-square* (*rms*) voltage and current as:

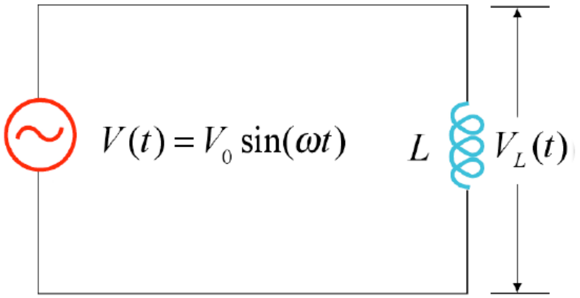
$$U_{rms} = \sqrt{\left\langle U_R^2\right\rangle} = \frac{U_{R0}}{\sqrt{2}}\ ;\ I_{rms} = \sqrt{\left\langle I_R^2\right\rangle} = \frac{I_{R0}}{\sqrt{2}}$$

The rms voltage supplied to the domestic wall outlets in Romania is 220 V for f=50 Hz
The amplitude of the AC voltage is $220 \times \sqrt{2} = 311.1\ V$

The power dissipated by the resistor: $P_R(t) = U_R(t)I_R(t) = I_R^2(t)R\ [W]$

The average power over one period is: $\boxed{\left\langle P_R(t)\right\rangle = \left\langle I_R^2(t)\right\rangle R = \frac{1}{2}I_{R0}^2 R = I_{rms}^2 R = U_{rms}I_{rms} = \frac{U_{rms}^2}{R}}$

8.4.3. Purely inductive load in AC



We can write:

$$U(t) = U_0 \sin(\omega \cdot t) = U_L$$

$$I_L(t) = I_{L0} \sin(\omega \cdot t + \varphi_L)$$

and $U_L = L\dfrac{dI_L}{dt}$

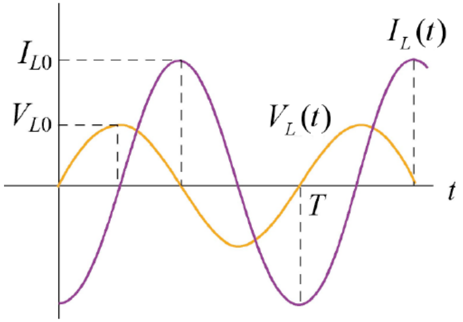
L – inductor with inductance L

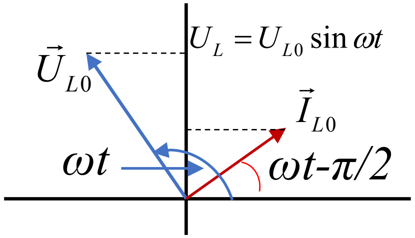
Like in section 8.4.2 we can write:
$$\begin{cases} U(t) - L\dfrac{dI_L}{dt} = 0 \Rightarrow \dfrac{dI_L}{dt} = \dfrac{U(t)}{L} = \dfrac{U_{L0}}{L}\sin\omega t \\ U_0 = U_{L0} \end{cases}$$

By integrating the above eq. we find: $I_L = \int dI_L = \dfrac{U_{L0}}{L}\int \sin\omega t dt = -\dfrac{U_{L0}}{L\omega}\cos\omega t$

$$\boxed{I_L(t) = \frac{U_{L0}}{L\omega}\sin(\omega t - \pi/2)} \Rightarrow \begin{cases} \varphi_L = -\dfrac{\pi}{2} \\ I_{L0} = \dfrac{U_{L0}}{\omega L} = \dfrac{U_{L0}}{X_L} \end{cases}$$

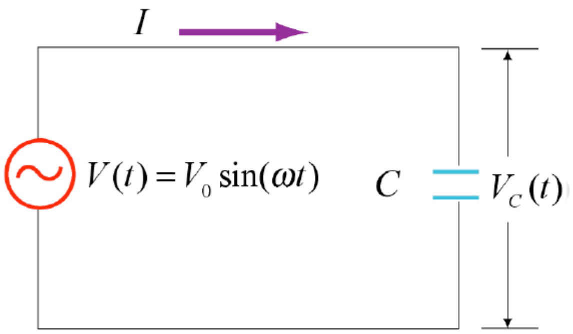
X_L is named *inductive reactance*
$X_L = \omega L \quad [\Omega]$





The current lags voltage by π/2 in a purely inductive circuit

8.4.4. Purely capacitive load in AC



We can write:

$$U(t) = U_0 \sin(\omega \cdot t) = U_C$$

$$I_C(t) = I_{C0} \sin(\omega \cdot t + \varphi_C)$$

and $U_C = \dfrac{Q}{C}$

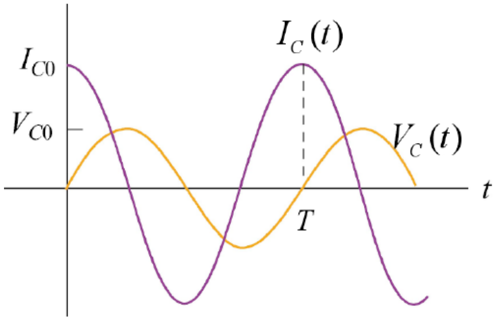
- we can write: $U(t) - U_C(t) = 0 \Rightarrow U(t) = \dfrac{Q(t)}{C} \Rightarrow Q(t) = C \cdot U(t) = C \cdot U_{C0} \sin \omega t$

$$U_0 = U_{C0}$$

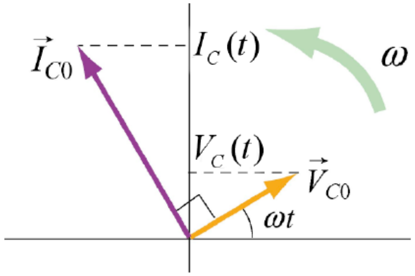
As $I_C(t) = \dfrac{dQ(t)}{dt} \Rightarrow I_C(t) = \omega \cdot C \cdot U_{C0} \cos \omega t = \omega \cdot C \cdot U_{C0} \sin(\omega t + \pi/2) \quad \Rightarrow \quad \varphi = \pi/2$

$$I_C(t) = I_{C0} \sin(\omega t + \pi/2) \text{ with } I_{C0} = U_{C0} \cdot \omega \cdot C = \frac{U_{C0}}{X_C}, \; X_C = \frac{1}{\omega C} \; [\Omega]$$

X_C is the capacitive reactance



(a)

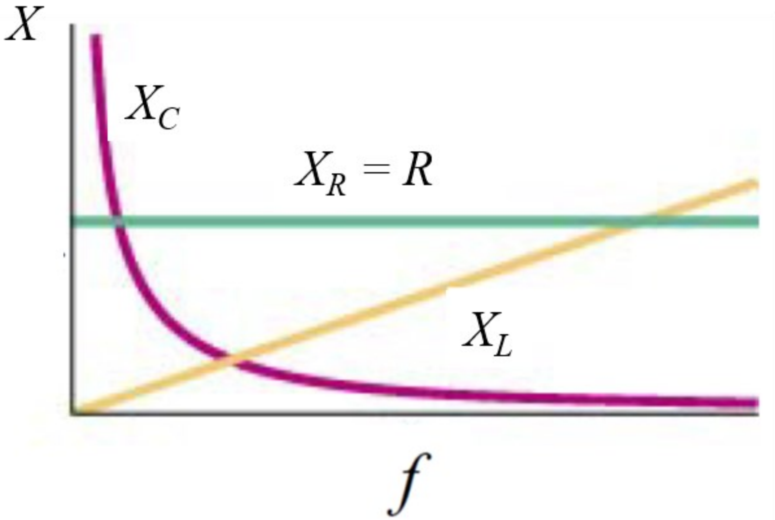


(b)

The current leads the voltage by π/2 in a purely capacitive circuit

Bellow, we can summarize the results from Sections 8.4.2, 8.4.3 and 8.4.4

Simple Circuit	R	L	C	$X_L = \omega L$	$X_C = \dfrac{1}{\omega C}$	φ (between U and I)
purely resistive	R	0	∞	0	0	0
purely inductive	0	L	∞	X_L	0	$\pi / 2$
purely capacitive	0	0	C	0	X_C	$-\pi / 2$

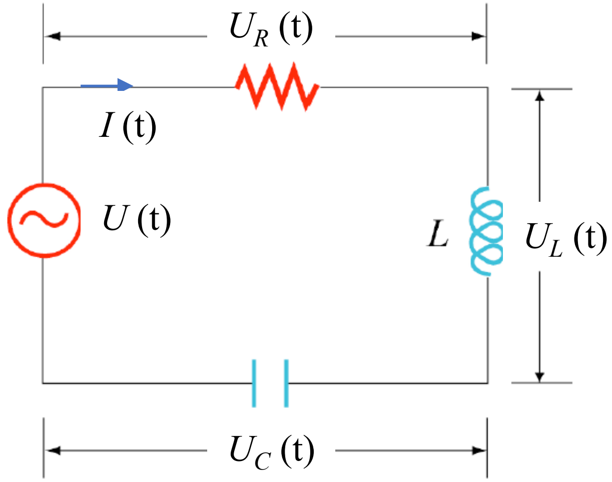


$$f = \frac{\omega}{2\pi}$$

Behaviour of R, L and C with frequency

8.4.5. AC driven RLC series circuit

Consider the circuit:



with: $I(t) = I_0 \sin(\omega \cdot t)$

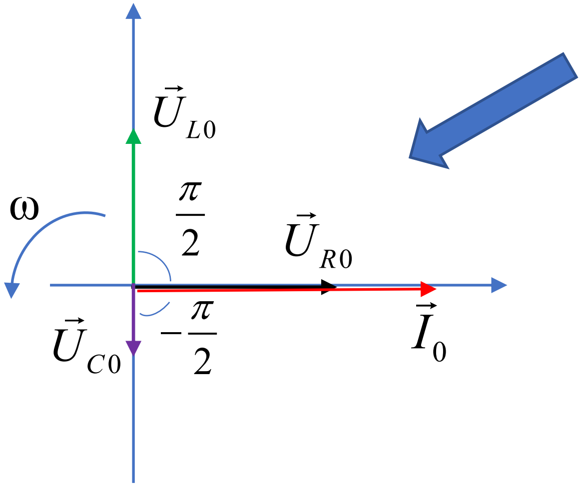
$U(t) = U_0 \sin(\omega \cdot t + \varphi)$

φ – phase difference between *I* and *U*

We can write:

$$\left\{ \begin{aligned}
&U_R(t) = I(t)R = I_0 R \sin \omega t = U_{R0} \sin \omega t \,;\; U_{R0} = I_0 R \\
&U_L(t) = L\frac{dI}{dt} = L\omega \cdot I_0 \cos \omega t = L\omega \cdot I_0 \sin\left(\omega t + \frac{\pi}{2}\right) \\
&U_L(t) = U_{L0} \sin\left(\omega t + \frac{\pi}{2}\right), \quad \text{with } U_{L0} = I_0 X_L \;;\; X_L = \omega L \\
&U_C(t) = \frac{Q(t)}{C}, \ \text{with } Q(t) = \int I(t)\,dt = I_0 \int \sin(\omega t) \cdot dt \\
&Q(t) = -\frac{1}{\omega} I_0 \cos \omega t = \frac{1}{\omega} I_0 \sin\left(\omega t - \frac{\pi}{2}\right) \\
&U_C(t) = \frac{Q(t)}{C} = \frac{1}{\omega C} I_0 \sin\left(\omega t - \frac{\pi}{2}\right)
\end{aligned} \right.$$

The phasor representation of signals:



Based on the previous findings, we can write:

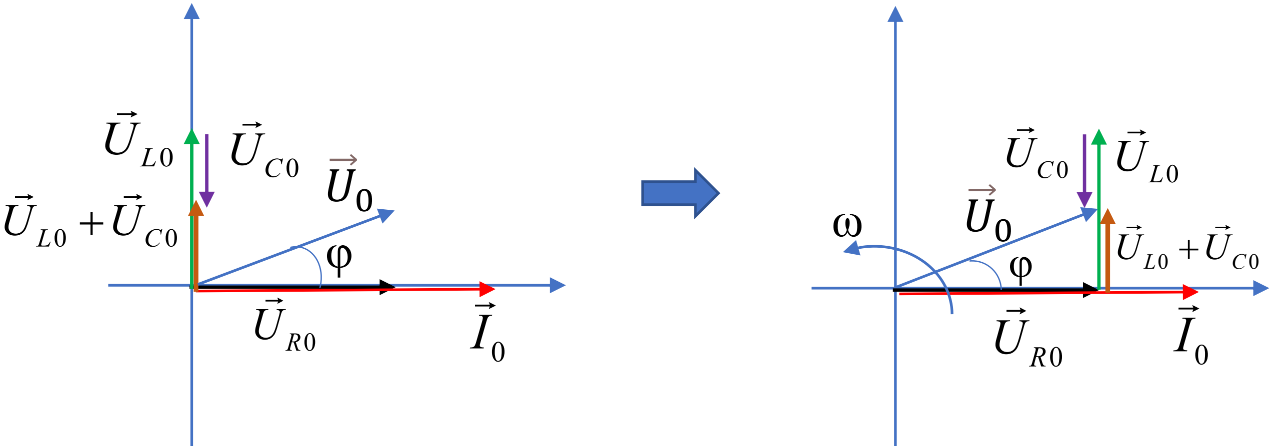
$$U_R(t)+U_L(t)+U_C(t)=U_0\sin(\omega t+\varphi)$$

$$RI_0\sin\omega t+L\omega\cdot I_0\sin\left(\omega t+\frac{\pi}{2}\right)+\frac{1}{\omega C}I_0\sin\left(\omega t-\frac{\pi}{2}\right)=U_0\sin(\omega t+\varphi)$$

$$U_{R0}\sin\omega t+U_{L0}\sin\left(\omega t+\frac{\pi}{2}\right)+U_{C0}\sin\left(\omega t-\frac{\pi}{2}\right)=U_0\sin(\omega t+\varphi)$$

with $U_{R0}=I_0R,\ U_{L0}=I_0L\omega,\ U_{C0}=I_0\dfrac{1}{\omega C}$ amplitudes of signals on R, L, C

$$\vec{U}_{R0}+\vec{U}_{L0}+\vec{U}_{C0}=\vec{U}_0$$



The phasor representation of signals:

In these plots:

$X_L > X_C \Rightarrow U_L > U_C$

- <u>inductive regime</u>; φ>0

If:

$X_L < X_C \Rightarrow U_L < U_C$

- <u>capacitive regime</u>; φ<0

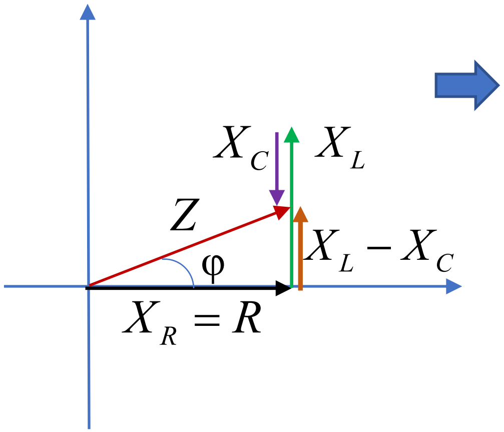
Now, we can express:

$$\vec{U}_{R0}+\vec{U}_{L0}+\vec{U}_{C0}=\vec{U}_0 \Rightarrow U_0=\sqrt{U_{R0}^2+\left(U_{L0}-U_{C0}\right)^2} \Rightarrow U_0=I_0\sqrt{R^2+\left(X_L-X_C\right)^2}$$

$$I_0=\frac{U_0}{\sqrt{R^2+\left(X_L-X_C\right)^2}}=\frac{U_0}{Z},\quad Z=\sqrt{R^2+\left(X_L-X_C\right)^2}$$

Z is named <u>impedance of the RLC series circuit</u>

<u>We can plot:</u>

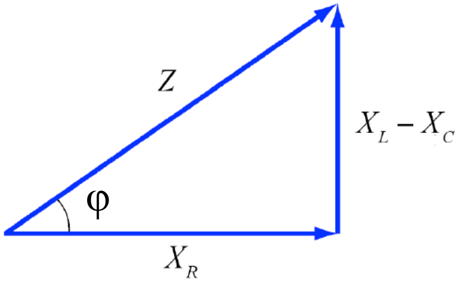


$$tg\varphi=\frac{X_L-X_C}{R}$$

$$\cos\varphi=\frac{R}{Z}$$ *<u>power factor</u>*

$$I_0=\frac{U_0}{\sqrt{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2}}=f\left(\omega\right),\quad Z=\sqrt{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2}$$

Vector representation of the relationship between Z , X_R , X_L , and X_C .



$$I\left(t\right)=\frac{U_0}{Z}\sin\omega t\ ;\ Z=\frac{R}{\cos\varphi} \Rightarrow I\left(t\right)=\frac{U_0}{R}\cos\varphi\sin\omega t$$

8.4.6. Resonance for RLC series circuit

We found that:

$$I_0 = \frac{U_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} = f(\omega), \quad Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

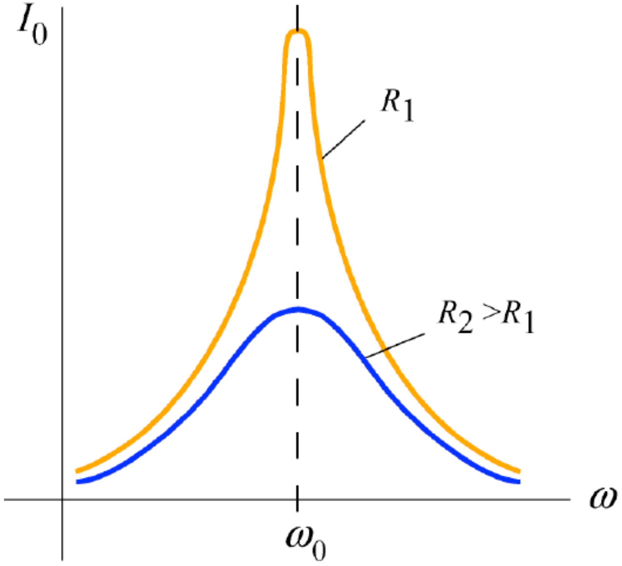
$$\text{if } \omega = 0 \Rightarrow I_0 \to 0$$

$$\text{if } \omega \to \infty \Rightarrow I_0 \to 0$$

We can find a pulsation, ω_0 (or frequency f_0) for which I_0 is maximum, i.e., *Z is minimum.*

ω_0 is named *resonance pulsation* and is found from:

$$\boxed{\omega L = \frac{1}{\omega C} \Rightarrow \omega_r \equiv \omega_0 = \frac{1}{\sqrt{LC}}}$$



$I_0=f(\omega)$ for two different values of the resistance

At resonance, $Z=R$ and $I_0=U_0/R$

$$tg\varphi = \frac{X_L - X_C}{R} = 0$$

$$\cos\varphi = \frac{R}{Z} = 1 \Rightarrow \varphi = 0$$

At resonance, I is in phase with U, i.e., φ=0

At resonance, the current amplitude is larger for a smaller value of resistance.

8.4.7. Power in AC for series RLC circuit

a) Power in AC – general relations

Having expressions of U and I, we can express the instantaneous power as:

$$I(t) = I_0 \sin(\omega \cdot t)$$

$$U(t) = U_0 \sin(\omega \cdot t + \varphi)$$

$$P(t) = U(t) \cdot I(t) = U_0 I_0 \left[\sin \omega t \cdot \left(\sin \omega t \cdot \cos \varphi + \cos \omega t \cdot \sin \varphi\right)\right] =$$

$$= \frac{U_0^2}{Z}\left[\sin \omega t \cdot \left(\sin \omega t \cdot \cos \varphi + \cos \omega t \cdot \sin \varphi\right)\right]$$

$$P(t) = \frac{U_0^2}{Z}\left[\underset{\substack{>0 \\ \text{Active power}}}{\sin^2 \omega t \cdot \cos \varphi} + \underset{\substack{>0 \text{ or } <0 \\ \text{Reactive power}}}{\frac{1}{2}\sin 2\omega t \cdot \sin \varphi}\right]$$

The mean power (time average of power):

$$\langle P \rangle \equiv \overline{P} = \frac{1}{T}\int_0^T P(t)\,dt = \frac{1}{2}U_0 I_0 \cos \varphi = \frac{1}{2}\frac{U_0^2}{Z}\cos \varphi$$

because:

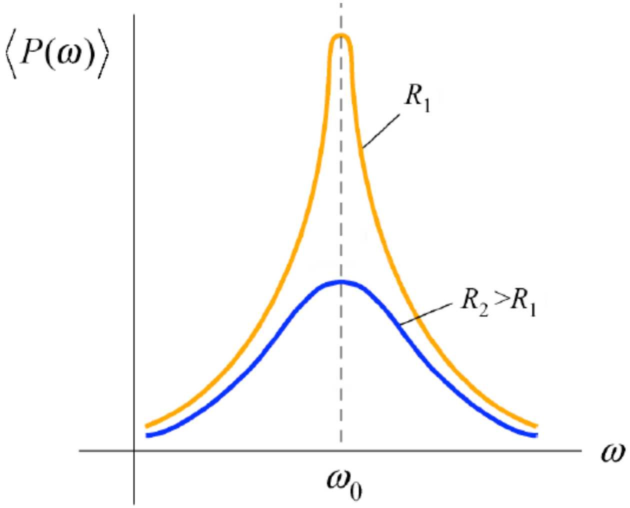
$$\int_0^T \sin 2\omega t \cdot dt = 0$$

$$\frac{1}{T}\int_0^T \sin^2 \omega t \cdot dt = \frac{1}{2}$$

$$\langle P\rangle = \frac{1}{2}U_0 I_0 \cos\varphi = \frac{1}{\sqrt{2}}U_0 \cdot \frac{1}{\sqrt{2}} I_0 \cos\varphi = U_{rms} \cdot I_{rms} \cdot \cos\varphi$$

with: $U_{rms} = \frac{1}{\sqrt{2}}U_0$; $I_{rms} = \frac{1}{\sqrt{2}} I_0$ - *rms*=**r**oot **m**ean **s**quare also known as the *effective values* for voltage and current

Because: $\cos\varphi = \frac{R}{Z}$ ⇨ $\langle P\rangle = U_{rms} \cdot I_{rms} \cdot \frac{R}{Z}$



Average power as a function of frequency in a driven series *RLC* circuit

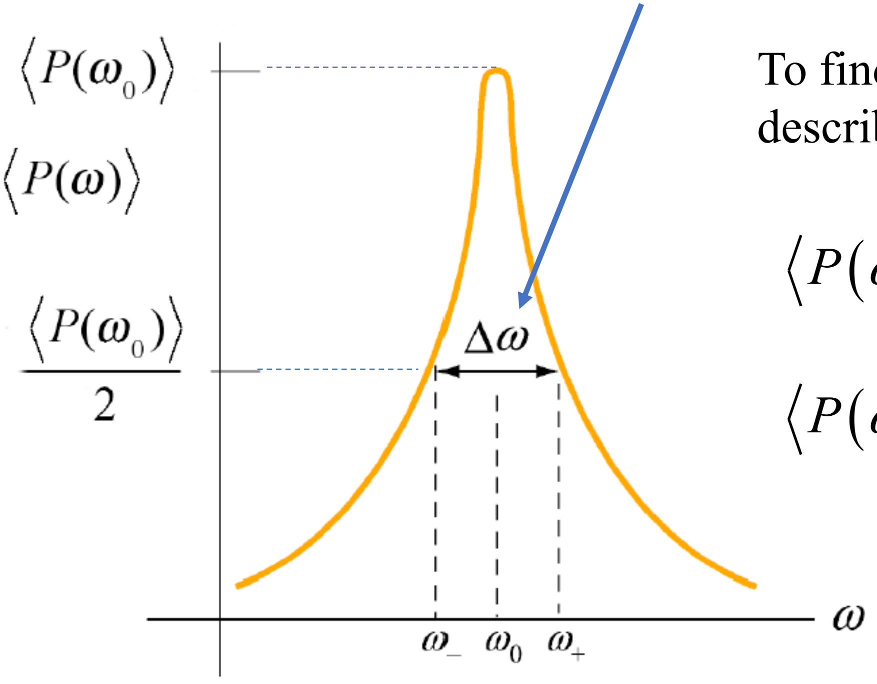
<P> attains the maximum value when cosφ =1, or $Z = R$. $Z = R$ represents the resonance condition.

At resonance we have:

$$\langle P(\omega_0)\rangle = U_{rms} \cdot I_{rms} = \frac{U_{rms}^2}{R}$$

b. Width of the Peak

One way to characterize the width is to define $\Delta\omega = \omega_+ - \omega_-$ where $\omega_{+,-}$ are the values of the driving angular frequency such that the power *is equal to half its maximum power at resonance.* This is called ***full width at half maximum***



To find define $\Delta\omega$ we must follow some steps like described bellow:

$$\langle P(\omega)\rangle \equiv \bar{P} = \frac{1}{2}\frac{U_0^2}{Z}\cos\varphi = \frac{1}{2}\frac{U_0^2}{Z^2}R \qquad \cos\varphi = \frac{R}{Z}$$

$$\langle P(\omega)\rangle = \frac{1}{2}\frac{U_0^2}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}R$$

$$\Downarrow$$

$$\langle P(\omega_0)\rangle = \frac{1}{2}\frac{U_0^2}{R}$$ - the power at resonance

By using $\omega_0 = \frac{1}{\sqrt{LC}} \Rightarrow C = \frac{1}{L\cdot\omega_0^2}$ we find: $\langle P(\omega)\rangle = \frac{1}{2}\frac{U_0^2\,\omega^2\,R}{\omega^2R^2 + L^2\left(\omega^2 - \omega_0^2\right)^2}$

The condition for finding ω± is:

$$\left\langle P(\omega_{\pm})\right\rangle = \frac{1}{2}\left\langle P(\omega_0)\right\rangle \Rightarrow \frac{1}{2}\left(\frac{1}{2}\frac{U_0^2}{R}\right) = \frac{1}{2}\frac{U_0^2\,\omega_{\pm}^2\,R}{\omega_{\pm}^2 R^2 + L^2\left(\omega_{\pm}^2-\omega_0^2\right)^2}$$

By solving the above equation, we find:

$$\left(\omega_{\pm}^2-\omega_0^2\right)^2 = \left(\frac{R\omega_{\pm}}{L}\right)^2 \Rightarrow \begin{cases} \text{for } \omega_+ > \omega_0 \Rightarrow \omega_+^2-\omega_0^2 = \dfrac{R\omega_{\pm}}{L} \Rightarrow \omega_+ = \dfrac{R}{2L} + \sqrt{\dfrac{R^2}{4L^2}+\omega_0^2} \\ \text{for } \omega_- < \omega_0 \Rightarrow \omega_-^2-\omega_0^2 = -\dfrac{R\omega_{\pm}}{L} \Rightarrow \omega_- = -\dfrac{R}{2L} + \sqrt{\dfrac{R^2}{4L^2}+\omega_0^2} \end{cases}$$

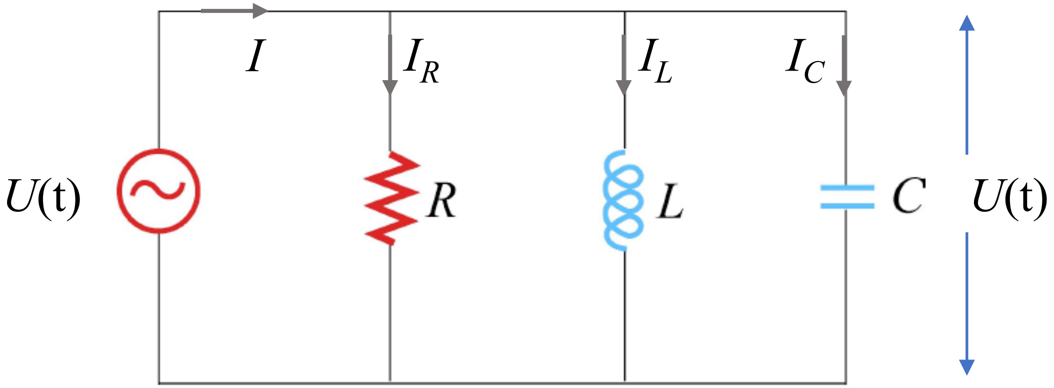
- these are solution with physical meaning

The width at half maximum is: $\boxed{\Delta\omega = \omega_+ - \omega_- = \frac{R}{L}}$ <u>*Δω increases with R*</u>

<u>The *quality factor*</u> Q_{qual} , or simply, *Q*, is defined as: $\boxed{Q = \frac{\omega_0}{\Delta\omega} = \frac{\omega_0 L}{R}}$

<u>*Q* decreases when *R* increases</u>

8.5. Parallel RLC *circuit*



Unlike the series *RLC* circuit, the instantaneous voltages across all three circuit elements *R* , *L* , and *C* are the same, and each voltage is in phase with the current through the resistor.

<u>However, the currents through each element will be different:</u> I_R , I_L and I_C.

The following equations can be used to analyse this circuit:

$$U(t) = U_0 \sin \omega t$$

$$U_R = U_L = U_C = U(t); \quad I(t) = I_R(t) + I_R(t) + I_C(t)$$

$$I(t) = I_0 \sin(\omega t + \varphi)$$

$$I_R(t) = \frac{U_0}{R} \sin \omega t \Rightarrow I_{R0} = \frac{U_0}{R}$$

➡ **The current through R is in phase with U**

The voltage across the inductor is:

$$U_L = L\frac{dI_L}{dt} \Rightarrow dI_L = \frac{1}{L}U_L dt = \frac{1}{L}U_0 \sin\omega t \cdot dt \Rightarrow I_L(t) = \frac{1}{L}U_0 \int \sin\omega t \cdot dt$$

$$I_L(t) = -\frac{1}{\omega L}U_0 \cos\omega t = \frac{1}{\omega L}U_0 \sin\left(\omega t - \frac{\pi}{2}\right) \Rightarrow \boxed{I_L(t) = I_{L0}\sin\left(\omega t - \frac{\pi}{2}\right) \quad \text{with} \quad I_{L0} = \frac{1}{\omega L}U_0}$$

We denote with $\boxed{X_L = \omega L}$ the inductive reactance ➡ $\boxed{I_{L0} = \frac{U_0}{X_L}}$

The current through inductor lags the voltage with π/2

The voltage across the capacitor is:

$$U_C(t) = \frac{Q}{C} \Rightarrow I_C(t) = \frac{dQ}{dt} = C\frac{dU_C(t)}{dt} = C \cdot U_0 \frac{d(\sin\omega t)}{dt} = \omega C \cdot U_0 \cos\omega t = \omega C \cdot U_0 \sin\left(\omega t + \frac{\pi}{2}\right)$$

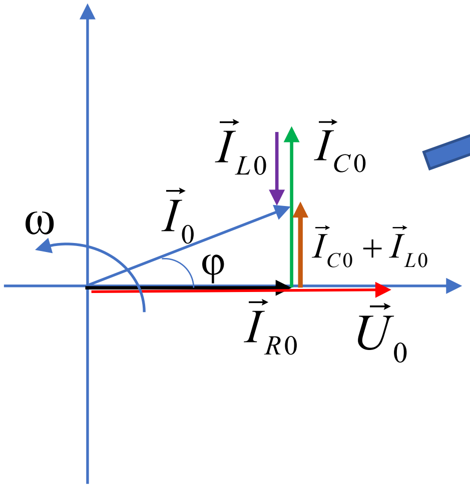
We denote with $\boxed{X_C = \frac{1}{\omega C}}$ the capacitive reactance ➡ $\boxed{I_{C0} = \frac{U_0}{X_C}}$

$$\boxed{I_C(t) = \frac{U_0}{X_C}\sin\left(\omega t + \frac{\pi}{2}\right) = I_{C0}\left(\omega t + \frac{\pi}{2}\right)}$$

The current through capacitor leads the voltage with π/2

Now, we can express the total current as:

$$I(t)=I_R(t)+I_L(t)+I_C(t)=I_{R0}\sin\omega t+I_{L0}\sin\left(\omega t-\frac{\pi}{2}\right)+I_{C0}\sin\left(\omega t+\frac{\pi}{2}\right)=I_0\sin(\omega t+\varphi)$$

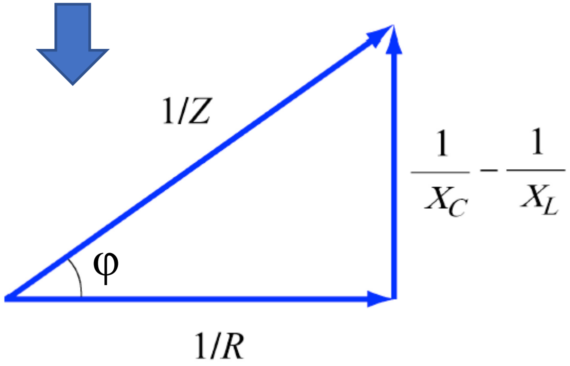


$$\vec{I}_0=\vec{I}_{R0}+\vec{I}_{L0}+\vec{I}_{C0}\Rightarrow I_0^2=I_{R0}^2+\left(I_{C0}-I_{L0}\right)^2$$

$$I_0=U_0\sqrt{\frac{1}{R^2}+\left(\frac{1}{X_C}-\frac{1}{X_L}\right)^2}=\frac{U_0}{Z}\Rightarrow\boxed{\frac{1}{Z}=\sqrt{\frac{1}{R^2}+\left(\omega C-\frac{1}{\omega L}\right)^2}}$$

$$tg\varphi=\frac{I_{C0}-I_{L0}}{I_{R0}}=\frac{\frac{1}{X_{C0}}-\frac{1}{X_{L0}}}{\frac{1}{R}}=R\left(\omega C-\frac{1}{\omega L}\right)$$

$$\cos\varphi=\frac{Z}{R}$$



Relationship between Z, R, X_L and X_C in a parallel *RLC* circuit

When $X_L=X_C$ ➔ 1/Z is minimum ➔ Z has a maximum value ➔ Z=R

The resonance condition for the parallel *RLC* means $1/X_L = 1/X_C$ ➡

$$\boxed{\begin{array}{l}\omega_r \equiv \omega_0 = \dfrac{1}{\sqrt{LC}} \\ \cos\varphi = 1 \text{ i.e., } \varphi = 0\end{array}}$$

The current in the inductor exactly cancels out the current in the capacitor, so that the current in the circuit reaches a minimum, and is equal to the current in the resistor

$$I_0 = \frac{U_0}{R}$$

As in the series *RLC* circuit, at resonance, power is dissipated only through the resistor

The time averaged power can be expressed as:

$$P(t) = U_0 I_0 (\sin\omega t)\left[\sin(\omega t + \varphi)\right] \Rightarrow \langle P(t)\rangle = \frac{1}{T}\int_0^T P(t)\cdot dt \Rightarrow \langle P\rangle = \frac{1}{2}U_0 I_0 \cos\varphi$$

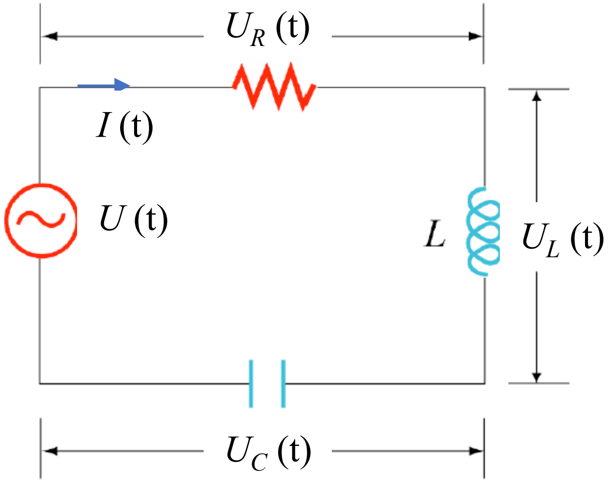
$$\text{but } I_{R0} = I_0\cos\varphi \text{ and } I_{R0} = \frac{U_0}{R} \Rightarrow \langle P\rangle = \frac{1}{2}\frac{U_0^2}{R} = \frac{1}{2}\frac{U_0^2}{Z}\cos\varphi$$

$$\text{with } \cos\varphi = \frac{Z}{R} = 1/\sqrt{1+\left(\omega RC - \frac{R}{\omega L}\right)^2}$$

8.6. Complex representation of electrical quantities

Instead of proceeding with the analysis of electrical circuits using the methods described in previous sections, we shall introduce an elegant technique often used in electrical engineering which consists in representation of Q(t), I(t) and U(t) as **complex quantities**. As we'll see, this approach will simplify the analysis of complex circuits.

Let's consider, again, the RLC series circuit:



$$U(t) = U_0 \sin \omega t \,;\; U_0 - \text{is the voltage amplitude}$$

$$U(t) = U_R(t) + U_L(t) + U_C(t)$$

$$\Downarrow$$

$$L\frac{dI}{dt} + I \cdot R + \frac{Q}{C} = U_0 \sin \omega t$$

We define the following complex quantities:

The complex voltage $\hat{U}(t) = U_0 \cdot e^{i\omega t} \Rightarrow \operatorname{Im}\{\hat{U}(t)\} = U_0 \sin \omega t$ is the physical quantity, $i = \sqrt{-1}$

The complex current is $\hat{I}$, while the complex charge is denoted with $\hat{Q}$.

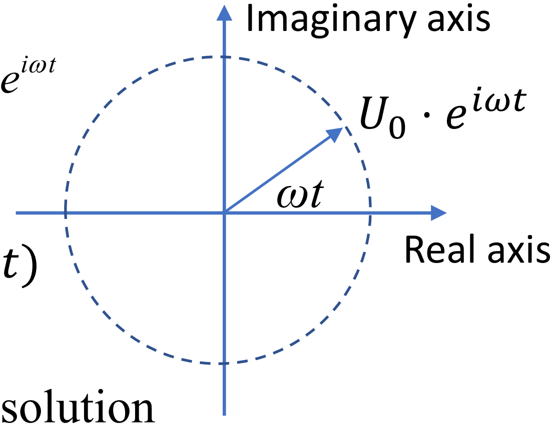
From: $I(t) = \dfrac{dQ}{dt} \Rightarrow \hat{I}(t) = \dfrac{d\hat{Q}}{dt} \Rightarrow I(t) = \operatorname{Im}\left\{\dfrac{d\hat{Q}}{dt}\right\} = \dfrac{d}{dt}\operatorname{Im}\{\hat{Q}\} = \dfrac{dQ}{dt}$

Now, the equation:

$$L\frac{dI}{dt}+I\cdot R+\frac{Q}{C}=U_0\sin\omega t$$

can be expressed as: $L\frac{d\hat{I}}{dt}+\hat{I}\cdot R+\frac{\hat{Q}}{C}=U_0e^{i\omega t} \Rightarrow L\frac{d^2\hat{I}}{dt^2}+\frac{d\hat{I}}{dt}\cdot R+\frac{1}{C}\frac{d\hat{Q}}{dt}=i\omega\cdot U_0e^{i\omega t}$

but $\frac{d\hat{Q}}{dt}=\hat{I} \Rightarrow L\frac{d^2\hat{I}}{dt^2}+R\cdot\frac{d\hat{I}}{dt}+\frac{1}{C}\hat{I}=i\omega\cdot U_0e^{i\omega t}$



$\hat{I}(t)$ has the same temporal variation like the driving voltage $\hat{U}(t)$

$$\boxed{\hat{I}(t)=\hat{I}_0e^{i\omega t}}$$ represents the steady state solution

$\hat{I}_0$ is named ***complex amplitude*** and contains the amplitude and the phase difference, φ, between the driving voltage and the current

$$\left.\begin{aligned}&L\frac{d^2\hat{I}}{dt^2}+R\cdot\frac{d\hat{I}}{dt}+\frac{1}{C}\hat{I}=i\omega\cdot U_0e^{i\omega t}\\&\frac{d\hat{I}}{dt}=i\omega\cdot\hat{I}_0e^{i\omega t}=i\omega\cdot\hat{I}\\&\frac{d^2\hat{I}}{dt^2}=-\omega^2\hat{I}_0e^{i\omega t}\end{aligned}\right\}\Rightarrow \hat{I}_0\left(-\omega^2L+i\omega R+\frac{1}{C}\right)e^{i\omega t}=i\omega\cdot U_0e^{i\omega t} \Rightarrow \hat{I}_0=\frac{U_0}{R+i\left(\omega L-\frac{1}{\omega C}\right)}$$

From $\hat{I}_0 = \dfrac{U_0}{R + i\left(\omega L - \dfrac{1}{\omega C}\right)} \Rightarrow \hat{Z} = R + i\left(X_L - X_C\right) = R + iX = Ze^{i\varphi}$ $\hat{Z}$ - ***complex impedance***

$$\Downarrow$$

$$\cos\varphi = \frac{R}{Z}$$

$$\tan\varphi = \frac{X}{R} \text{ and } \left|\hat{Z}\right| = Z = \sqrt{R^2 + X^2}$$

with $X = X_L - X_C$

$$\hat{Z} = \hat{Z}_R + \hat{Z}_L + \hat{Z}_C$$

$$\Downarrow$$

$$\hat{I}_0 = \frac{U_0}{Ze^{i\varphi}} = \frac{U_0}{Z}e^{-i\varphi} = I_0 e^{-i\varphi} \text{ with } I_0 = \frac{U_0}{Z}$$

$$\hat{I}\left(t\right) = I_0 e^{i(\omega t - \varphi)} \Rightarrow \text{physical solution: } I\left(t\right) = \operatorname{Im}\left\{\hat{I}\left(t\right)\right\} = \frac{U_0}{Z}\sin\left(\omega t - \varphi\right) = I_0 \sin\left(\omega t - \varphi\right)$$

$$\hat{I} \cdot \hat{Z} = \hat{U}\text{, with } \hat{Z} = R + i\left(X_L - X_C\right) \Rightarrow \hat{I}\left(R + i\omega L - i\frac{1}{\omega C}\right) = \hat{U}$$

But $i = e^{i\frac{\pi}{2}}$ and -$i = e^{-i\frac{\pi}{2}} \Rightarrow \hat{I}R + \hat{I} \cdot X_L e^{i\frac{\pi}{2}} + \hat{I} \cdot X_C e^{-i\frac{\pi}{2}} = \hat{U}$

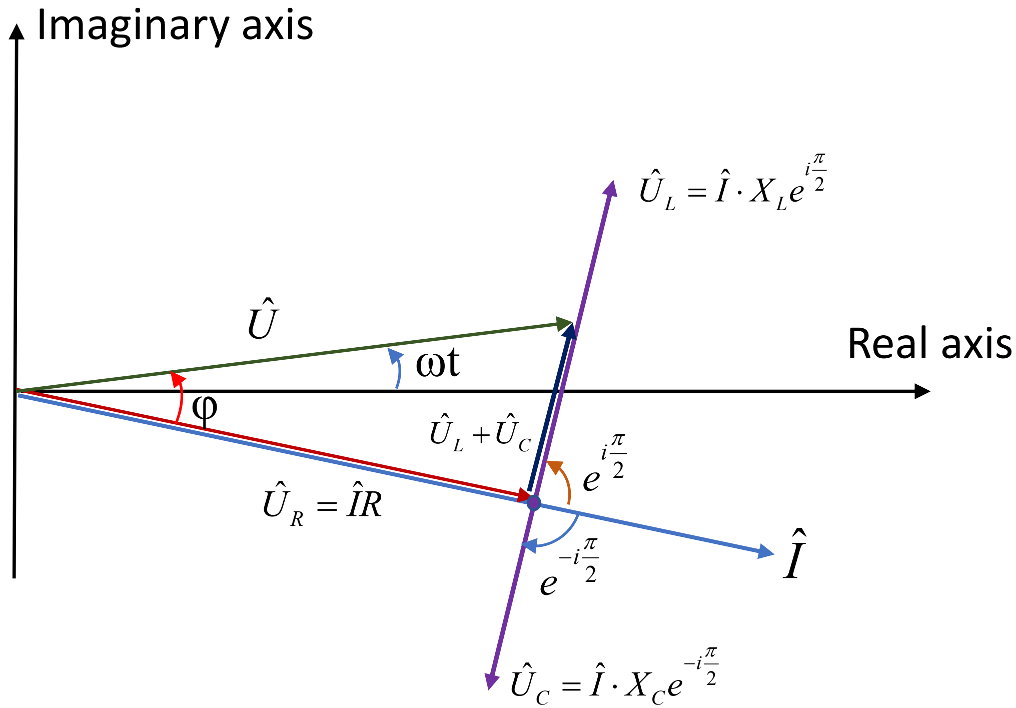
with $\hat{U}_R = \hat{I}R$, $\hat{U}_L = \hat{I} \cdot X_L e^{i\frac{\pi}{2}}$ and $\hat{U}_C = \hat{I} \cdot X_C e^{-i\frac{\pi}{2}} \Rightarrow \hat{U} = \hat{U}_R + \hat{U}_L + \hat{U}_C$

From:

$$\hat{I}R+\hat{I}\cdot X_L e^{i\frac{\pi}{2}}+\hat{I}\cdot X_C e^{-i\frac{\pi}{2}}=\hat{U}\text{, with } \hat{U}=U_0 e^{i\omega t}$$

$$\text{with } \hat{U}_R=\hat{I}R,\ \hat{U}_L=\hat{I}\cdot X_L e^{i\frac{\pi}{2}} \text{ and } \hat{U}_C=\hat{I}\cdot X_C e^{-i\frac{\pi}{2}} \Rightarrow \hat{U}=\hat{U}_R+\hat{U}_L+\hat{U}_C$$

we can plot the following Argand diagram for a series RLC circuit:



Argand diagram for a series RLC circuit

We can define the following:

$\frac{1}{Z}=G$ – Admittance [Ω^{-1}]=[S]

$\frac{1}{R}=C$ – Conductance [Ω^{-1}]=[S]

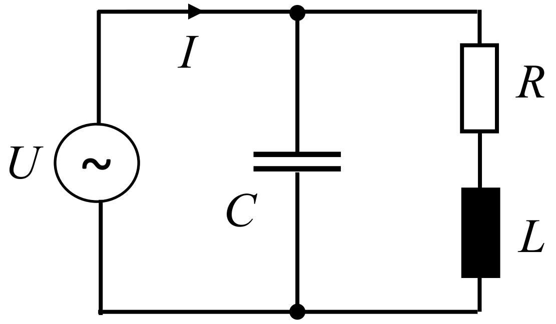
$\frac{1}{X}=Y$ – Susceptance [Ω^{-1}]=[S]

[S] - Siemens

$$\hat{Z}_R=R$$
$$\hat{Z}_L=iX_L=i\omega L$$
$$\hat{Z}_C=-iX_C=-i\frac{1}{\omega C}$$

8.6.1. Example of a simple AC network

Given the circuit bellow, find the resonance pulsation, ω_r.



$$\hat{Z}_1 = -iX_C = -\frac{i}{\omega C}$$

$$\hat{Z}_2 = \hat{Z}_R + \hat{Z}_L = R + i\omega L$$

$$\Downarrow$$

$$\frac{1}{\hat{Z}} = \frac{1}{\hat{Z}_1} + \frac{1}{\hat{Z}_1} \Rightarrow \hat{Z} = \frac{\hat{Z}_1 \cdot \hat{Z}_2}{\hat{Z}_1 + \hat{Z}_2} = \frac{-\frac{i}{\omega C}\left(R + i\omega L\right)}{-\frac{i}{\omega C} + R + i\omega L}$$

<u>At resonance</u>, φ=0 ➔ $Im\{\hat{Z}\} = 0$

By doing basic calculations with complex numbers, we can separate the real part from the imaginary part, and we find:

$$\hat{Z} = \frac{R + i\left(\omega L - \omega R^2 C - \omega^3 L^2 C\right)}{\omega^2 C^2 \left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)\right]} \Rightarrow \omega_r L - \omega_r R^2 C - \omega_r^3 L^2 C = 0 \quad \text{such that } \cos\varphi = 1$$

The equation bellow can be simplified with ω_r because $\omega_r > 0$.

$$\omega_r L - \omega_r R^2 C - \omega_r^3 L^2 C = 0 \;\Rightarrow\; L - R^2 C - \omega_r^2 L^2 C = 0$$

$$\Downarrow$$

$$\boxed{\omega_r^2 = \frac{1}{LC} - \frac{R^2}{L^2}}$$

We know that:

$\omega_0^2 = \dfrac{1}{LC}$ the resonance pulsation for a pure LC circuit; $Q = \dfrac{L}{R}\dfrac{1}{\sqrt{LC}} = \dfrac{\omega_0 L}{R}$ the quality factor

From basic calculations we find:

$$\omega_r^2 = \frac{1}{LC} - \frac{R^2}{L^2} = \omega_0^2 - \frac{CR^2}{LC \cdot L} = \omega_0^2 - \omega_0^2 \frac{CR^2}{L} \;;\; \frac{CR^2}{L} = \frac{1}{\omega_0^2}\frac{R^2}{L^2} = \frac{1}{Q^2}$$

$$\omega_r^2 = \omega_0^2 - \omega_0^2 \cdot Q^{-2} \;\Rightarrow\; \boxed{\omega_r = \omega_0\sqrt{1 - Q^{-2}} \;;\; \omega_r \simeq \omega_0 \text{ when } Q \gg 1}$$

8.7. Power in an AC circuit

8.7.1. Power in AC – general relations

Having expressions of U and I, we can express the instantaneous power as:

$$I(t) = I_0 \sin(\omega \cdot t)$$

$$U(t) = U_0 \sin(\omega \cdot t + \varphi)$$

$$P(t) = U(t) \cdot I(t) = U_0 I_0 \left[\sin \omega t \cdot (\sin \omega t \cdot \cos \varphi + \cos \omega t \cdot \sin \varphi)\right] =$$

$$= \frac{U_0^2}{Z}\left[\sin \omega t \cdot (\sin \omega t \cdot \cos \varphi + \cos \omega t \cdot \sin \varphi)\right]$$

$$P(t) = \frac{U_0^2}{Z}\left[\sin^2 \omega t \cdot \cos \varphi + \frac{1}{2}\sin 2\omega t \cdot \sin \varphi\right]$$

>0 Active power

>0 or <0 Reactive power

The mean power (time average of power):

$$\langle P \rangle \equiv \overline{P} = \frac{1}{T}\int_0^T P(t)\,dt = \frac{1}{2}U_0 I_0 \cos \varphi = \frac{1}{2}\frac{U_0^2}{Z}\cos \varphi$$

because:

$$\int_0^T \sin 2\omega t \cdot dt = 0$$

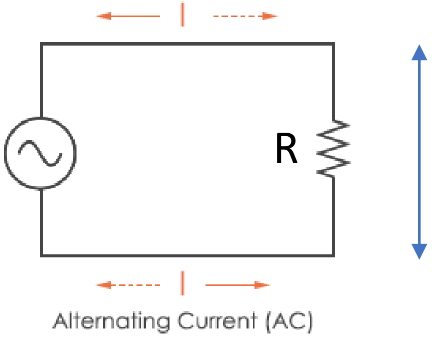
$$\frac{1}{T}\int_0^T \sin^2 \omega t \cdot dt = \frac{1}{2}$$

In a simple alternating current (AC) circuit consisting of a source and a linear load, both the current and voltage are sinusoidal.

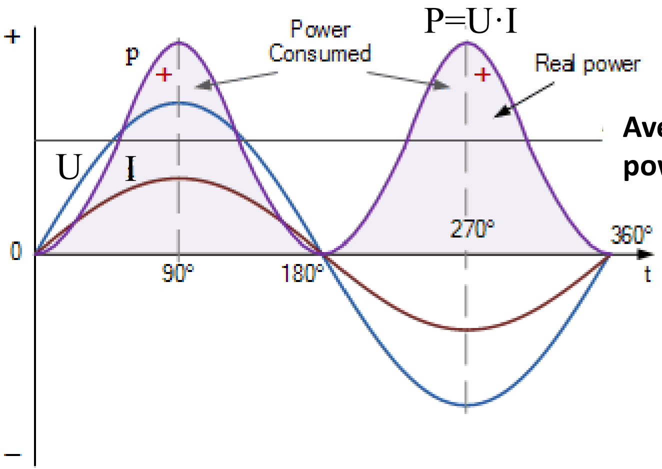
a) If the load is purely resistive, the two quantities reverse their polarity at the same time. At every instant the product of voltage and current is positive or zero, the result being that the direction of energy flow does not reverse.
In this case, only **active power** is transferred.

$$I(t) = I_{R0} \sin(\omega \cdot t)$$

$$U(t) = U_0 \sin(\omega \cdot t)$$



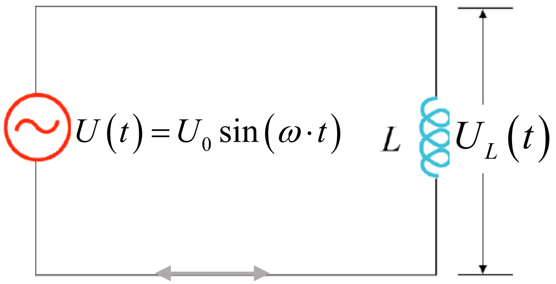
$$U_R(t)$$



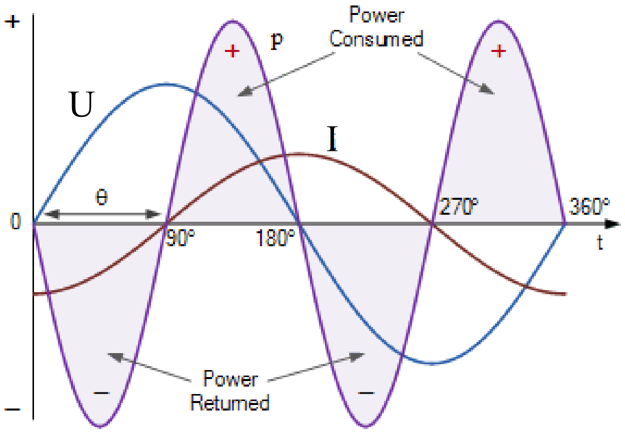
Average power >0

Image adapted from https://www.electronics-tutorials.ws/accircuits/power-in-ac-circuits.html

b) As we shown previously, in a **purely inductive circuit**, the current always “lags” behind the voltage by 90º (π/2) as shown bellow:

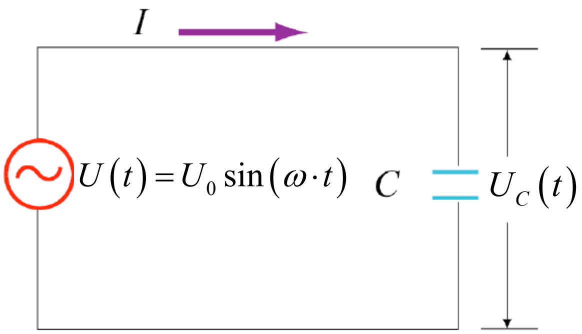


$$I_L(t)=\frac{U_{L0}}{L\omega}\sin(\omega t-\pi/2)$$

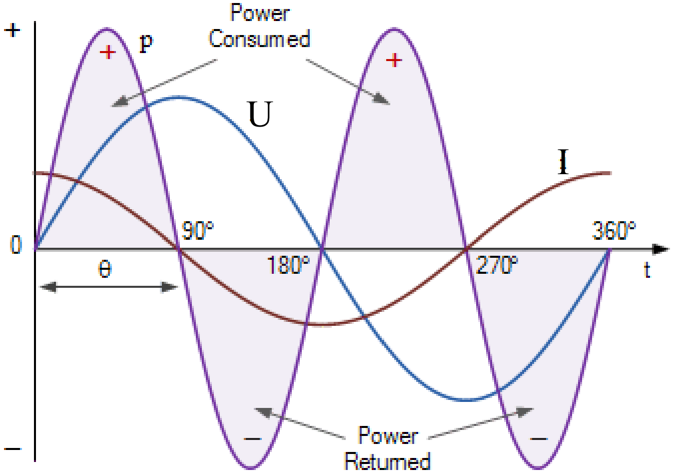


Average power =0

c) In a **purely capacitive circuit**, the current always “leads” the voltage by 90º (π/2) as shown bellow:



$$I_C(t)=\frac{U_{C0}}{X_C}\sin(\omega t+\pi/2)$$



Average power =0

Image adapted from https://www.electronics-tutorials.ws/accircuits/power-in-ac-circuits.html

In conclusion:

If the load is purely reactive, then the **voltage and current are 90 degrees out of phase**. For two quarters of each cycle, the product of voltage and current is positive, but for the other two quarters, the product is negative, indicating that on average, exactly as much energy flows into the load as flows back out.

There is no net energy flow over each half cycle. In this case, **only reactive power flows**: there is no net transfer of energy to the load.

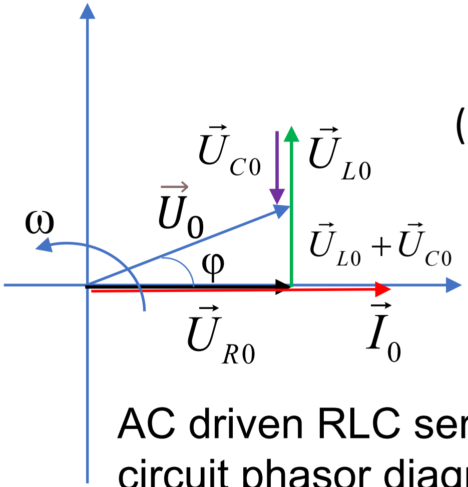
However, electrical power does flow along the wires and returns by flowing in reverse along the same wires. The current required for this reactive power flow dissipates energy in the line resistance, even if the ideal load device consumes no energy itself. Practical loads have resistance as well as inductance, or capacitance, so both active and reactive powers will flow to normal loads.

Apparent power is the product of the rms values of voltage and current.

The active power, represents **power dissipation as heat in resistor** and $P>0$

8.7.2. **Power triangle**

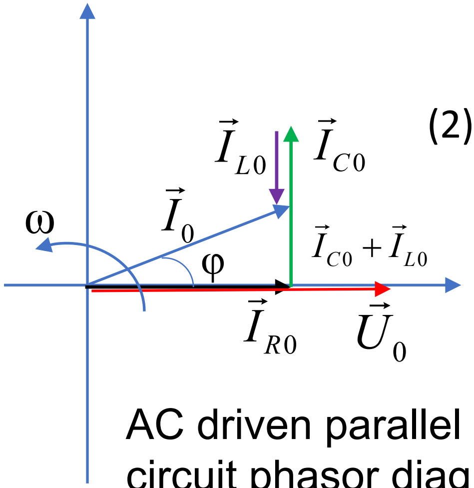
From the previous sections we found the following diagrams for voltages and currents:



(1)

AC driven RLC series circuit phasor diagram

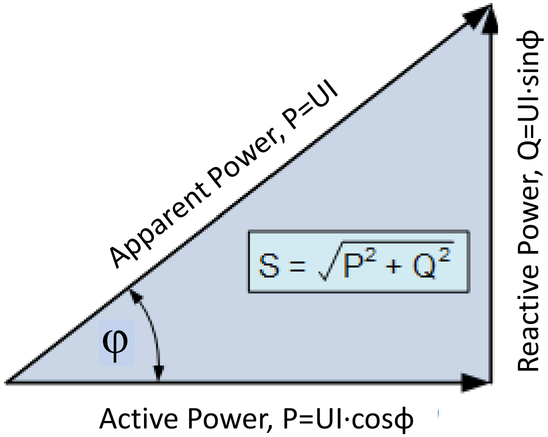
and:



(2)

AC driven parallel RLC circuit phasor diagram

By multiplying with I, diagram (1) or by U, diagram (2) we get the **Power triangle**:

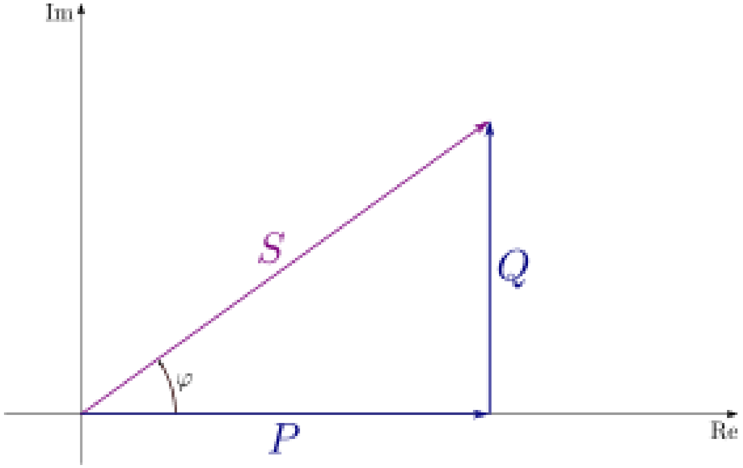


The relationship of the three elements of power:

- P - active power, (watts) – W; P>0, represents **power dissipation as heat in resistor**
- S - apparent power, (VA)
- Q - reactive power, (VAR); Q can be either >0 or <0.

Reactive power is related with the electric energy needed to sustain the electric and magnetic fields required by alternating current equipment which contains L and C.

8.7.3. The complex power – complex quantities approach



The complex power is the vector sum of active and reactive power. The apparent power is the magnitude of the complex power.

$$\hat{S} = P + iQ$$

Active power, *P,* (W)
Reactive power, *Q,* (VAR)
Complex power, $\hat{S}$, (VA)
Apparent power, $|\hat{S}|$, (VA) (the magnitude of $\hat{S}$)

The apparent power is:

$\hat{S} = \hat{U} \cdot \hat{I}^*$ where $\hat{I}^*$ is the complex conjugate of $\hat{I}$

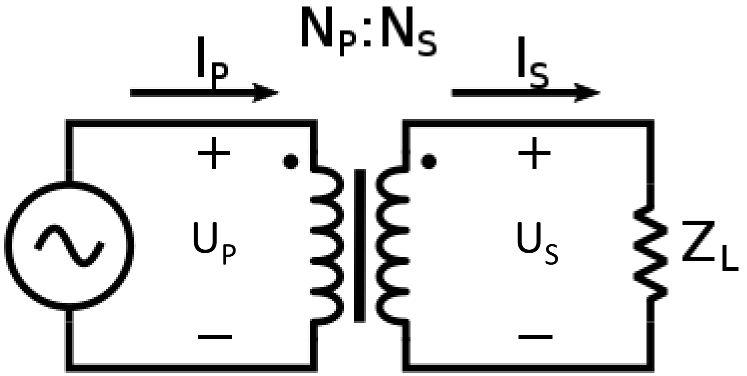
$\hat{S} = P + i \cdot Q$ and $|S| = \sqrt{P^2 + Q^2}$

$$S = \left|\hat{I}\right|^2 \cdot \hat{Z} = \frac{\left|\hat{U}\right|^2}{\hat{Z}^*} \quad P = \frac{\left|\hat{U}\right|^2}{R}$$

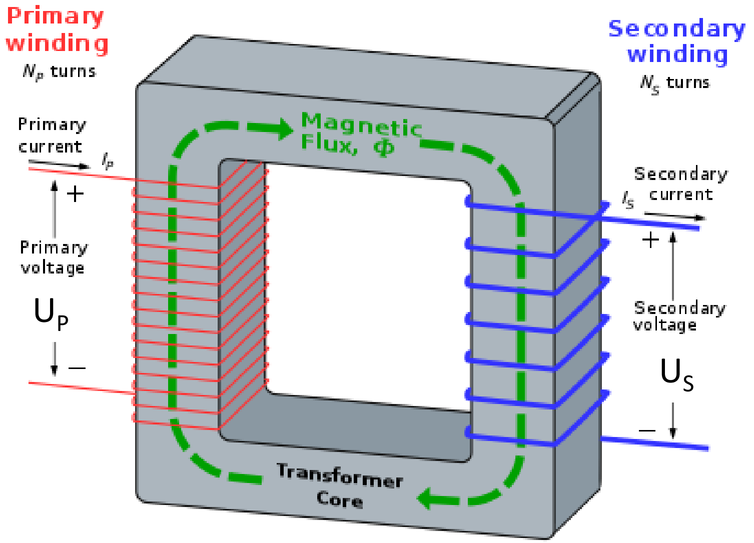
and $P = |S| \cdot \cos\varphi$; $Q = |P| \cdot \sin\varphi$

8.8. Electric transformer

A transformer is a passive electrical device based on electromagnetic induction law applied to coupled inductors. The main role of the transformer is to transfer, in a convenient way, electrical energy from one electrical circuit to another, or multiple circuits.



Ideal transformer model



If:

$U_P > U_S$ – step down transformer

$U_P < U_S$ – step up transformer

a) <u>There is no load</u>: $Z \to \infty$

Assume ideal conditions for primary and secondary coils: $R_P=0$ and $C_1=0$, $R_S=0$ and $C_S=0$

$Z_P = X_{Lp} = \omega L_P$ - the power supply source is connected to a pure inductive reactance

$$tg\varphi_P = \frac{X_{Lp}}{R_p} \to \infty \Rightarrow \varphi_P = \frac{\pi}{2} \Rightarrow \cos\varphi_P = 0$$ ➡ <u>There is no consumption of energy</u>

We know that:

$\varphi_M = B_P \cdot A$ - the magnetic flux inside the magnetic core; B_P is produced by I_p.

$\Downarrow$

$\Phi_P = N_P \varphi_M = N_P \cdot B_P \cdot A$ - the flux in the primary coil

$\Phi_S = N_S \varphi_M = N_S \cdot B_P \cdot A$ - the flux in the secondary coil

➡

$$U_P(t) = -\frac{d\Phi_P}{dt} = -N_P \frac{d\varphi_M}{dt}$$

$$U_S(t) = -\frac{d\Phi_S}{dt} = -N_S \frac{d\varphi_M}{dt}$$

We can write for an ideal transformer:

$$\frac{U_P}{U_S} = \frac{N_P}{N_S} \Rightarrow \frac{U_P}{N_P} = \frac{U_S}{N_S} \Rightarrow \frac{U_{Prms}}{N_P} = \frac{U_{Srms}}{N_S}$$

➡

$N_1 > N_2$ – step down transformer

$N_1 < N_2$ – step up transformer

b) There is a load: $Z \neq 0$

$I_S(t) \neq 0; \langle P_S \rangle = U_{S\,rms} I_{S\,rms} \cos\varphi \neq 0$

Usually, $Z_{LS} \approx X_{LS} = \omega L_S \gg R_S$

I_S induces a back e.m.f. in secondary coil which opposes voltage U_P ➔ it needs a current to flow in the primary coil to sustain I_S .

By the law of conservation of energy, apparent, real and reactive power are each conserved in the input and output: $U_P I_P = U_S I_S$.

Such that we can write: $\frac{U_P}{U_S} = \frac{I_S}{I_P} = \frac{N_P}{N_S} = \sqrt{\frac{L_P}{L_S}}$

8.9. Proposed and solved applications

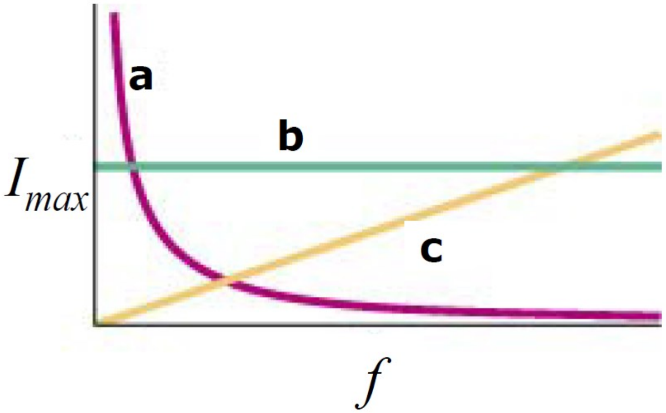
1. Two solenoids with $N_1=10$ and $N_2=1000$ turns are wrapped on the same cylindrical core such that they have the same length $l=10$ cm and the transversal surface of the core is $S=1$ cm^2. The core consists of vacuum $\mu_0=4\pi*10^{-7}$ (H/m). A current $I_1=I_0 \sin(\omega t)$ flows through coil 1, where $I_0=1$ A and the current's frequency is $\nu=1000/(2\pi)$ Hz.
Find the mutual inductance M_{21} of solenoid 2 with respect to 1. Also find the induced voltage E_{21} in solenoid 2.

2. Calculate the self-inductance of a long solenoid, with length $l=5$ cm and a radius $R=1$ cm, that consists of $N=100$ turns of wire; neglect the end effects. The core consists of vacuum $\mu_0=4\pi*10^{-7}$ (H/m)

3. Calculate the self-inductance of a toroidal coil that has the inner radius $R_i=4$ cm and the external radius $R_e=5$ cm. The solenoid has a rectangular cross section with height $h=1$ cm. The coil has $N=200$ turns and the core consists from air ($\mu_{air}\approx\mu_0=4\pi*10^{-7}$ H/m).

4. Calculate the mutual inductance, M, between a toroid, like from problem 3, and a linking coil with $N_2=10$ turns (see the draw). The linking coil has a cross section larger than the toroid's cross section.

5. Three identical EMF sources are hooked to a single circuit element: a resistor, a capacitor, or an inductor. The current amplitude is then measured as a function of frequency. Which one of the following curves corresponds to an inductive circuit? Discuss the solution.



6. A generator produces current at a frequency of 60 Hz with peak voltage and current amplitudes of 100V and 10A, respectively. What is the average power produced if they are in phase?

Short instructions:

$$P_{ave} = \frac{1}{2}U_0 I_0 \cos\varphi \text{ ; here, } \cos\varphi = 0 \Rightarrow P_{ave} = \frac{1}{2}U_0 I_0 = \ldots$$

7. A RLC series circuit, with R = 200 Ω, C = 15 μF, L = 230 mH, is connected to an A.C. voltage $U=U_0\sin\omega t$ where U_0=36 V and f = 60 Hz. Find: a) the resonant frequency, f_0, b) X_L and X_C, c) the circuit impedance, Z, d) the current amplitude, I_0 and e) the phase difference between current and voltage.

8. A RLC series circuit, with R = 200 Ω, X_C=150 Ω, X_L=80 Ω, is connected to an A.C. voltage U_{rms}=120 V and f = 60 Hz. Find: a) the circuit impedance, Z, b) the effective value of the current, c) the current amplitude, d) the power factor and e) the average power.
How much capacitance must be added to maximize the power in the circuit (and thus bring it into resonance)?

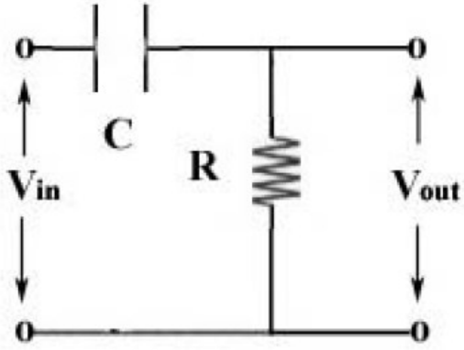
9. A variable frequency EMF source with U_{rms}=6 V is connected to a resistor and inductor series connection; R=80 Ω and L=40 mH.
a) At what frequency, f, does $U_R = U_L$? b) At that frequency, what is phase angle φ?
c) What is the current amplitude? and d) What is the rms current?

Short instructions:

a) $R=\omega L \Rightarrow 2\pi f\cdot L = R \Rightarrow f = ...$ b) $tg\varphi = \frac{X_L}{R} = 1 \Rightarrow \varphi = \frac{\pi}{4} = 45°$

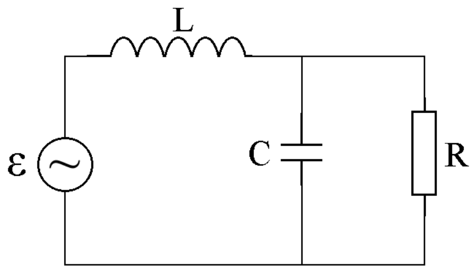
c) $I_0 = \frac{\sqrt{2}\cdot U_{rms}}{\sqrt{R^2+\omega^2 L^2}} = \frac{\sqrt{2}\cdot U_{rms}}{R\sqrt{2}} = \frac{U_{rms}}{R} = ...$ because $R = X_L$ d) $I_{rms} = \frac{I_0}{\sqrt{2}} = ...$

10. Consider a circuit consisting of a resistor and a capacitor with an AC sinusoidal input $U_{in}(t)=U_0\sin(\omega t)$ and two output terminals like in the figure bellow.
Consider $U_0 = 10$ V, C=1 μF, R=1 kΩ and ω=1000 rad/s.



a) What is the impedance of this circuit?
b) What is the amplitude and the phase of the current in the circuit?
c) What is the amplitude and the phase of the output voltage $U_{out}(t)$ accros the resistor?
d) What is the ratio of the magnitudes of the output signal amplitude to the input signal amplitude $|U_{out\ 0}|/|U_{in\ 0}|$?
e) Explain why this type of circuit is referred to as a "*high pass filter*".

11. The circuit below has R=1 kΩ and C=100 nF and is connected to an alternating voltage source with ω=1000 rad/s. Calculate L such that the current through L will be in phase with voltage (φ=0).

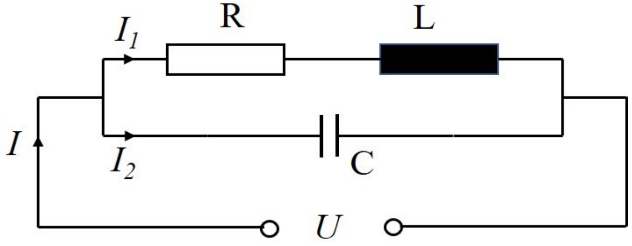


$$\hat{Z} = \hat{Z}_L + \hat{Z}_{CR} = i\omega L + \frac{-\frac{i}{\omega C}R}{R - \frac{i}{\omega C}} \text{ and from } \operatorname{Im}\left\{\hat{Z}\right\} = 0$$

$$\Rightarrow L = \frac{R^2 C}{1 + R^2\omega^2 C^2} = \ldots$$

12. Consider the circuit from application 7 where R=1 kΩ, L=10 μH and C=100 nF, which is connected to an alternating voltage source. Calculate ω such that the current through L will be in phase with voltage (φ=0).

13. The circuit bellow is driven by an AC power supply source with frequency f= 50 Hz and the effective voltage U=220 V; R=10 Ω and L=0.1 H. Calculate:
a) C such that I is in phase with U (φ=0); b) I_1, I_2 and I when φ=0.



a) $$\hat{Z} = \frac{\hat{Z}_{LR}\cdot\hat{Z}_C}{\hat{Z}_{LR}\cdot\hat{Z}_C} = \frac{(R+iX_L)(-iX_C)}{R+iX_L-iX_C} = \frac{X_LX_C - iRX_C}{R+i(X_L-X_C)}$$

$X = X_L - X_C$ - effective reactance

$$\varphi = 0 \Rightarrow \operatorname{Im}\{\hat{Z}\} = 0 \Rightarrow X_C = \frac{R^2+X_L^2}{X_L} \Rightarrow C = \ldots$$

b) $$\hat{I}_{LR} = \frac{\hat{U}_{eff}}{\hat{Z}_{LR}} = \frac{\hat{U}_{eff}}{R+iX_L} = \frac{\hat{U}_{eff}}{R+i\cdot\omega L} = \frac{220}{10+i\cdot 31.4} = 2.02 - i\cdot 6.36 \ \ [\text{A}]$$

$$I_1 = I_{LR} = \sqrt{2.02^2+3.36^2} = 6.673\ [\text{A}]$$

c) $$\hat{I}_C = \frac{\hat{U}_{eff}}{Z_C} = \frac{\hat{U}_{eff}}{-i\dfrac{1}{\omega C}} = i\cdot 6.36\ [\text{A}] \Rightarrow I_2 = 6.36\ [\text{A}]$$

14. A parallel RLC circuit is driven by an AC power supply source. Find the ratio R/X_L such that the ratio between the active power and the reactive power, $P_a/P_r=3/4$, and $X_C=4X_L$.

This is the end of the course…

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Appendix A

International System of Units

Quantity	**Name**	**Symbol**
SI Base Units		
Length	meter	m
Mass	kilogram	kg
Time	second	s
Electric current	ampere	A
Temperature	kelvin	K
Amount of substance	mole	mol
SI Derived Units with Special Names		
Frequency	Hertz	Hz (s^{-1})
Force	Newton	N ($kg·m/s^2$)
Pressure	Pascal	Pa (N/m^2)
Energy, work	Joule	J (N·m)
Power	Watt	W (J/s)
Electric charge	Coulomb	C (A·s)
Electric potential	Volt	V (W/A)
Capacitance	Farad	F (C/V)
Electric resistance	Ohm	Ω (V/A)
Conductance	Siemens	S (A/V)
Magnetic flux	Weber	Wb (V·s)
Magnetic flux density	Tesla	T (Wb/m^2)
Magnetic field strength		A/m
Inductance	Henry	H (Wb/A)
Magnetic moment	m	m ($A·m^2$)
Magnetization	M	M (A/m)

Appendix B

Some Physical Constants

Avogadro's number	N_0=6.02205x10^{23} mol^{-1}
Universal gaz constant	R=8.3144 J/mol·K
Boltzmann constant	k=1.38066x10^{-23} J/K
Stefan-Boltzmann constant	σ=5.6703x10^{-8} (W/m^2)/K^4
Coulomb constant	K=8.98755x10^9N·m^2/C^2
Permittivity constant (1/4πK)	ε_0=8.8542x10^{-12} C^2/N·m^2 or C/V·m, F/m
Permeability constant	μ_0=4π·10^{-7} N/A^2 or T·m/A, H/m
Velocity of light in vacuum	c=2.997925x10^8 m/s
Electron mass	m_e=9.1095x10^{-31} kg
	=5.4858x10^{-4} u
Proton mass	m_p=1.67265x10^{-27} kg
	=1.0072765 u
Neutron mass	m_n=1.67495x10^{-27} kg
	=1.0086650 u
Hydrogen atom mass	M_H=1.67356x10^{-27} kg
	=1.0078250 u
Rydberg constant	R_∞=1.097373x10^7 m^{-1}
Electron charge	e=1.60219x10^{-19} C
Planck constant	h=6.62618x10^{-34} J·s
	$\hbar$=h/2π=1.05459x10^{-34} J·s
Bohr radius	a_0=5.2918x10^{-11} m
Gravitation constant	γ=6.673x10^{-11} N·m^2/kg^2