ON COMPACTNESS VIA $bI$-OPEN SETS IN IDEAL BITOPOLUTIONAL SPACES

Santanu ACHARJEE*, Ivan L. REILLY 2 and Diganta Jyoti SARMA 3

Abstract

Sarma introduced the concept of $(i, j)$-$bI$-compact spaces with respect to an ideal in a bitopological space. In this paper, our aim is to present several characterizations of these spaces. Also, the class of $(i, j)$-$bI$-countably compact spaces is introduced with some of its properties.

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1 Introduction

The concept of bitopological space was introduced by Kelly [10] in 1963. It is universally accepted that compactness is one of the most important notions of topology and it has a very significant role in the theory of topological spaces. The concept of ideal was introduced by Kuratowski [11] and further it was investigated by Vaidyanathswamy [16], Jankovic and Hamlett [9] and many others. With the concept of ideal, Newcomb [12] defined the notions of compactness and countable compactness in topological spaces. These notions have been investigated by Hamlett and Jankovic [7] and Hamlett et al. [8].

According to Swart [15], a cover $\mathcal{U}$ of a bitopological space $(X, \tau_1, \tau_2)$ is called $\tau_1\tau_2$-open if $\mathcal{U} \subseteq \tau_1 \cup \tau_2$. If in addition, $\mathcal{U}$ contains at least one non-empty member of $\tau_1$ and at least one non-empty member of $\tau_2$, then it is called pairwise open [6]. If every pairwise open cover of $(X, \tau_1, \tau_2)$ has a finite subcover then the space is called pairwise compact [6]. $A \subseteq X$ of $(X, \tau_1, \tau_2)$ is said to be semi-open [5] if

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1* Corresponding author, Economics and Computational Rationality Group, Department of Mathematics Debraj Roy College, Assam, e-mail: sacharjee326@gmail.com, santanuacharjee@rediffmail.com

2Department of Mathematics, University of Auckland, New Zealand, e-mail: i.reilly@auckland.ac.nz

3Department of Mathematics, Central Institute of Technology, Assam, e-mail: dj.sarma@cit.ac.in, djs_math@rediffmail.com
it is open in the upper bound topology of \( \tau_1 \) and \( \tau_2 \). A set \( A \subseteq X \) of \((X, \tau_1, \tau_2)\) is said to be semi-compact \([5]\) if it is compact in the upper bound topology of \( \tau_1 \) and \( \tau_2 \); in other words \( A \) is semi-compact if and only if, given any covering of \( A \) by semi-open subsets of \( X \), there exists a finite subcovering.

According to Swart \([15]\), a cover \( \mathcal{U} \) of a bitopological space \((X, \tau_1, \tau_2)\) is called \( \tau_1\tau_2 \)-open if \( \mathcal{U} \subseteq \tau_1 \cup \tau_2 \). If in addition, \( \mathcal{U} \) contains at least one non-empty member of \( \tau_1 \) and at least one non-empty member of \( \tau_2 \); then it is called pairwise open \([6]\). If every pairwise open cover of \((X, \tau_1, \tau_2)\) has a finite subcover then the space is called pairwise compact \([6]\). A \( \subseteq X \) of \((X, \tau_1, \tau_2)\) is said to be semi-open \([5]\) if it is open in the upper bound topology of \( \tau_1 \) and \( \tau_2 \). A set \( A \subseteq X \) of \((X, \tau_1, \tau_2)\) is said to be semi-compact \([5]\) if it is compact in the upper bound topology of \( \tau_1 \) and \( \tau_2 \); in other words \( A \) is semi-compact if and only if, given any covering of \( A \) by semi-open subsets of \( X \), there exists a finite subcovering.

Reilly \([13]\) defined \((X, \tau_1, \tau_2)\) to be pairwise Lindelöf (respectively pairwise compact \([6]\)) if each pairwise open cover of \( X \) has a countable (respectively finite) subcover. Cooke and Reilly \([4]\) investigated the relationship between semi-compactness and pairwise compactness in a bitopological space.

Andrijevic \([3]\) introduced the notion of \( b \)-open set in a topological space and this notion has been extended to bitopological space by Al-Hawary and Al-Omari \([2]\). Sarma \([14]\) defined the notions of \((i, j)\)-\(b\)-I-open set, \((i, j)\)-\(b\)-I-continuous function and obtained several characterizations.

By \((i, j)\), bitopologists often mean the pair of topologies \((\tau_i, \tau_j)\) on various characterizations of a bitopological space \((X, \tau_1, \tau_2)\), where \( i, j \in \{1, 2\}, i \neq j \).

In this paper, we investigate various properties of \((\tau_1, \tau_2)\)-\(b\)-I-compactness in the bitopological category. The notion of \((\tau_1, \tau_2)\)-\(b\)-I-countably compactness is introduced and some properties are provided. The need to introduce these new notions can be understood by the fact that various generalized forms of compactness in advanced versions of general topology have been playing crucial roles in research related to theoretical physics, molecular topology, computer science, economics, biology, neuroscience, etc. Even after more than fifty years from its introduction by Kelly \([10]\), it can be easily found that bitopological results have not been used widely by researchers for the betterment of human race. Thus, with the views of application oriented theoretical developments of bitopological space, we introduce \((\tau_1, \tau_2)\)-\(b\)-I-compactness and \((\tau_1, \tau_2)\)-\(b\)-I-countably compactness.

2 Preliminaries

In this paper, \((X, \tau_1, \tau_2)\) denotes a bitopological space and \((X, \tau_1, \tau_2, I)\) denotes an ideal bitopological space, on which no separation axioms are assumed.

According to Kuratowski \([11]\), a collection \( I \subseteq P(X) \), where \( P(X) \) is the power set of a non-empty set \( X \) in a topological space \((X, \tau)\), is an ideal of \( X \) if it satisfies the following conditions:

(i) if \( Q \in I \) and \( R \subseteq Q \), then \( R \in I \)

(ii) if \( I \) and \( J \) are ideals of \( X \), then \( I \cap J \) is an ideal of \( X \).

(iii) the empty set \( \emptyset \) is an ideal of \( X \).

An ideal of \( X \) is called a \( \tau \)-ideal if it is an ideal of \((X, \tau)\), where \( \tau \) is a topology on \( X \).
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(ii) if $Q \in I$ and $R \in I$, then $Q \cup R \in I$.

For any ideal topological space $(X, \tau, I)$, the operator $(\cdot)^* : P(X) \to P(X)$ is the local function \[8\] of a subset $Q$ of $X$ with respect to the topology $\tau$ and ideal $I$; and it is defined as $Q^*(\tau, I) = \{a \in X : R \cap Q \notin I \text{ for every } R \in \tau(a)\}$, where $\tau(a) = \{U \in \tau : a \in U\}$. Simply, we write $Q^*$ instead of $Q^*(\tau, I)$ if there is no chance of confusion. Kuratowski closure operator for a topology $\tau^*(I) = \{U \subseteq X : cl^*(X - U) = X - U\}$ \[9\] on $X$ is defined by $cl^*(Q) = Q \cup Q^*$. Here, $\tau^*(I)$ is finer than $\tau$.

**Definition 2.1.** (2) A subset $P$ of a bitopological space $(X, \tau_1, \tau_2)$ is said to be $(i, j)$-$bI$-open if $P \subseteq i - \text{int}(j - \text{cl}(P)) \cup j - \text{cl}(i - \text{int}(P))$, where $i, j \in \{1, 2\}, i \neq j$.

The complement of $(i, j)$-$bI$-open set is called $(i, j)$-$bI$-closed set.

**Definition 2.2.** (14) A subset $P$ of an ideal bitopological space $(X, \tau_1, \tau_2, I)$ is said to be $(i, j)$-$bI$-open if $P \subseteq i - \text{int}(j - \text{cl}^*(P)) \cup j - \text{cl}^*(i - \text{int}(P))$, where $i, j \in \{1, 2\}, i \neq j$.

The complement of $(i, j)$-$bI$-open set is called $(i, j)$-$bI$-closed set.

**Remark 2.1.** (14) Every $(i, j)$-$bI$-open set is $(i, j)$-$bI$-closed.

**Definition 2.3.** (14) A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be $(i, j)$-$bI$-continuous if the inverse image of every $\sigma_i$-open set in $Y$ is $(i, j)$-$bI$-open in $X$, where $i, j = 1, 2$ and $i \neq j$.

**Definition 2.4.** (14) A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be $(i, j)$-$bI$-irresolute if the inverse image of every $(i, j)$-$bI$-open set in $Y$ is $(i, j)$-$bI$-open in $X$, where $i, j = 1, 2$ and $i \neq j$.

**Lemma 2.1.** (14) Let $(X, \tau_1, \tau_2, I)$ be an ideal bitopological space and $P, Q$ are subsets of $X$ such that $Q \subseteq P$. Then, $Q^*(\tau_i|_P, I|_P) = Q^*(\tau_i, I) \cap P$, for $i \in \{1, 2\}$.

Here, $\tau_i|_P$ is the relative topology on $P$ and $I|_P = \{P \cap M : M \in I\}$ is an ideal on $P$.

**Lemma 2.2.** (12) For any function $f : (X, \tau, I) \to (Y, \sigma)$, $f(I)$ is an ideal on $Y$.

**Lemma 2.3.** (12) If a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2, I)$ is an injection, then $f^{-1}(I) = \{f^{-1}(P) : P \in I\}$ is an ideal on $X$.

3 On $(\tau_1, \tau_2)$-$bI$-compactness

In this section, we introduce some new notions and study some properties.
Theorem 3.3. (14) An ideal bitopological space \((X, \tau_1, \tau_2, I)\) is said to be \((i, j)\)-I-compact, if for every cover \(\{P_\mu : \mu \in \Lambda\}\) by \(\tau_1\)-open sets of \(X\), there exists a finite subset \(\Lambda_0\) of \(\Lambda\) such that \(X \setminus \bigcup\{P_\mu : \mu \in \Lambda_0\} \in I\), where \(i, j = \{1, 2\}\) and \(i \neq j\).

In Definition 3.1 of [1], Acharjee and Tripathy have introduced some concepts of Lindelöfness in an ideal bitopological space.

Definition 3.2. (14) An ideal bitopological space \((X, \tau_1, \tau_2, I)\) is said to be \((i, j)\)-bl-compact, if for every cover \(\{P_\mu : \mu \in \Lambda\}\) by \((i, j)\)-bI-open sets of \(X\), there exists a finite subset \(\Lambda_0\) of \(\Lambda\) such that \(X \setminus \bigcup\{P_\mu : \mu \in \Lambda_0\} \in I\), where \(i, j = \{1, 2\}\) and \(i \neq j\).

An example of a \((\tau_1, \tau_2)\)-bI-compact space is given below.

Example 1. Let \(X = \{p, q, r\}, \tau_1 = \{\emptyset, \{p\}, X\}, \tau_2 = \{\emptyset, \{q\}, X\}\) and \(I = \{\emptyset, \{p\}\}\). Then, the set of \((\tau_1, \tau_2)\)-bI-open sets of \(X\) is \(\{\emptyset, \{p\}, \{q\}, \{p, q\}, \{q, r\}, X\}\) and \((X, \tau_1, \tau_2, I)\) is \((\tau_1, \tau_2)\)-bI-compact.

The proof of the following result is straight forward.

Theorem 3.1. Let \(I\) and \(J\) be two ideals in a bitopological space \((X, \tau_1, \tau_2)\) such that \(J \subseteq I\). If the space \((X, \tau_1, \tau_2, I)\) is \((\tau_1, \tau_2)\)-bI-compact, then \((X, \tau_1, \tau_2, I)\) is \((\tau_1, \tau_2)\)-bJ-compact.

Theorem 3.2. For a space \((X, \tau_1, \tau_2, I)\), the following are equivalent:

(i) \((X, \tau_1, \tau_2, I)\) is \((\tau_1, \tau_2)\)-bI-compact,

(ii) \((X, \tau_1^*, \tau_2^*, I)\) is \((\tau_1^*, \tau_2^*)\)-bI-compact.

Proof. (i) \(\Rightarrow\) (ii) Suppose \(\{P_\mu : \mu \in \Lambda\}\) is a cover of \(X\) by \((\tau_1^*, \tau_2^*)\)-bI-open sets of \(X\) such that \(P_\mu = Q_\mu \setminus J_\mu\), where \(Q_\mu = (\tau_1, \tau_2)\)-bI-open and \(J_\mu \in I\). Then, \(\{Q_\mu : \mu \in \Lambda\}\) is a cover of \(X\) by \((\tau_1, \tau_2)\)-bI-open sets and since, \((X, \tau_1, \tau_2, I)\) is \((\tau_1, \tau_2)\)-bI-compact, there exists a finite subset \(\Lambda_0\) of \(\Lambda\) such that \(X \setminus \bigcup\{Q_\mu : \mu \in \Lambda_0\} \in I\). Therefore, \(X \setminus \bigcup\{P_\mu : \mu \in \Lambda_0\} \subseteq [X \setminus \bigcup\{Q_\mu : \mu \in \Lambda_0\}] \cup [\bigcup\{J_\mu : \mu \in \Lambda_0\}] \in I\). Thus, \((X, \tau_1^*, \tau_2^*, I)\) is \((\tau_1^*, \tau_2^*)\)-bI-compact.

(ii) \(\Rightarrow\) (i) It is obvious, since \(\tau \subseteq \tau^*\).

Theorem 3.3. For a space \((X, \tau_1, \tau_2, I)\), the following are equivalent:

(i) \((X, \tau_1, \tau_2, I)\) is \((\tau_1, \tau_2)\)-bI-compact,
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Definition 3.3. A function $f : (X_1,\tau_1,\tau_2) \rightarrow (Y_1,\sigma_1,\sigma_2, f(I))$ is called $(\tau_1,\tau_2)$-strongly-$bI$-open if $f(P)$ is $(\sigma_1,\sigma_2)$-$bf(I)$-open in $Y_1$, for every $(\tau_1,\tau_2)$-$bI$-open set $P$ in $X_1$.

The proof of the next result parallels that of Theorem 3.4.

Theorem 3.5. Let $f : (X_1,\tau_1,\tau_2,f^{-1}(I)) \rightarrow (Y_1,\sigma_1,\sigma_2, I)$ be a $(\tau_1,\tau_2)$-strongly-$bI$-open bijection. If $Y_1$ is $(\sigma_1,\sigma_2)$-$bf(I)$-compact, then $(X_1,\tau_1,\tau_2,f^{-1}(I))$ is $(\tau_1,\tau_2)$-$bf^{-1}(I)$-compact.

Theorem 3.6. If a space $(X_1,\tau_1,\tau_2, I)$ is $(\tau_1,\tau_2)$-$bI$-compact and $\tau_i \cap I = \emptyset$ for $i \in \{1,2\}$, then there exists a finite collection $\{P_\mu : \mu \in \Lambda_0\}$ of $(\tau_1,\tau_2)$-$bI$-open sets of $X_1$ such that $X_1 = \tau_i - cl(\cup\{P_\mu : \mu \in \Lambda_0\})$ for $i \in \{1,2\}$.

Proof. Suppose that for any finite collection $\{P_\mu : \mu \in \Lambda_0\}$ of $(\tau_1,\tau_2)$-$bI$-open sets of $X_1$, $X_1 \neq \tau_i - cl(\cup\{P_\mu : \mu \in \Lambda_0\})$ for $i \in \{1,2\}$.

There exists $x \in X_1 \setminus \tau_i - cl(\cup\{P_\mu : \mu \in \Lambda_0\})$. Hence, there exits $U \in \tau_i$ containing $x$ such that $U \cap (\cup\{P_\mu : \mu \in \Lambda_0\}) = \emptyset$. Therefore, $U \subseteq X_1 \setminus (\cup\{P_\mu : \mu \in \Lambda_0\})$.
Definition 3.4. A subset $Q$ of an ideal bitopological space $(X, \tau_1, \tau_2, I)$ is said to be $(\tau_1, \tau_2)$-$bI$-compact relative to $X$ if for each cover $\{P_\mu : \mu \in \Lambda\}$ of $Q$ by $(\tau_1, \tau_2)$-$bI$-open sets of $X$, there exists a finite subset $\Lambda_0$ of $\Lambda$ such that $Q \setminus \bigcup\{P_\mu : \mu \in \Lambda_0\} \in I$.

Proof. Let a subset $Q$ be $(\tau_1, \tau_2)$-$bI$-compact relative to $X$ and $\{(P_\mu \cap Q) : \mu \in \Lambda\}$ be a cover of $Q$ by $(\tau_1|Q, \tau_2|Q)$-$bI|Q$-open sets in $Q$.

Then, $\{P_\mu : \mu \in \Lambda\}$ is a cover of $Q$ by $(\tau_1, \tau_2)$-$bI$-open sets of $X$. Since, $Q$ is $(\tau_1, \tau_2)$-$bI$-compact relative to $X$, thus there exists a finite subset $\Lambda_0$ of $\Lambda$ such that $M = Q \setminus \bigcup\{P_\mu : \mu \in \Lambda_0\} \in I$. Then, $Q \cap M \in I|Q$. This implies $Q \setminus \bigcup\{(P_\mu \cap Q) : \mu \in \Lambda_0\} \in I|Q$. Hence, $Q$ is $(\tau_1|Q, \tau_2|Q)$-$bI|Q$-compact.

Conversely, let $\{P_\mu : \mu \in \Lambda\}$ be a cover of $Q$ by $(\tau_1, \tau_2)$-$bI$-open sets of $X$. Thus, $\{(P_\mu \cap Q) : \mu \in \Lambda\}$ is a cover of $Q$ by $(\tau_1|Q, \tau_2|Q)$-$bI|Q$-open sets of $Q$. Hence, there exists a finite subset $\Lambda_0$ of $\Lambda$ such that $Q \setminus \bigcup\{(P_\mu \cap Q) : \mu \in \Lambda_0\} \subseteq I|Q \subseteq I$. Since $Q \setminus \bigcup\{P_\mu : \mu \in \Lambda_0\} \subseteq Q \setminus \bigcup\{(P_\mu \cap Q) : \mu \in \Lambda_0\} \in I$, $Q$ is $(\tau_1, \tau_2)$-$bI$-compact relative to $X$.

Definition 3.5. An ideal bitopological space $(X, \tau_1, \tau_2, I)$ is said to be $(\tau_1, \tau_2)$-$bI$-countably compact, if for each countable cover $\{P_\mu : \mu \in \Lambda\}$ of $X$ by $(\tau_1, \tau_2)$-$bI$-open sets of $X$, there exists a finite subset $\Lambda_0$ of $\Lambda$ such that $X \setminus \bigcup\{P_\mu : \mu \in \Lambda_0\} \in I$.

The following results are stated without proof.

Theorem 3.9. For a space $(X, \tau_1, \tau_2, I)$, the following are equivalent:

(i) $(X, \tau_1, \tau_2, I)$ is $(\tau_1, \tau_2)$-$bI$-countably compact,

(ii) if the countable collection $\{P_\mu : \mu \in \Lambda\}$ of $(\tau_1, \tau_2)$-$bI$-closed sets in $X$ is such that $\bigcap\{P_\mu : \mu \in \Lambda\} = \emptyset$, then there exists a finite subset $\Lambda_0$ of $\Lambda$ such that $\bigcap\{P_\mu : \mu \in \Lambda_0\} \in I$.

Theorem 3.10. If $f : (X, \tau_1, \tau_2, I) \to (Y, \sigma_1, \sigma_2, f(I))$ is a $(\tau_1, \tau_2)$-$bI$-irresolute surjection and $X$ is $(\tau_1, \tau_2)$-$bI$-countably compact, then $(Y, \sigma_1, \sigma_2, f(I))$ is $(\sigma_1, \sigma_2)$-$bf(I)$-countably compact.
Theorem 3.11. Let \( f : (X, \tau_1, \tau_2, f^{-1}(I)) \rightarrow (Y, \sigma_1, \sigma_2, I) \) be a \((\tau_1, \tau_2)\)-strongly-\(bI\)-open bijection. If \( Y \) is \((\sigma_1, \sigma_2)\)-\(bI\)-countably compact, then \((X, \tau_1, \tau_2, f^{-1}(I))\) is \((\tau_1, \tau_2)\)-\(bf^{-1}(I)\)-countably compact.

Conflict of interest: Authors declare that there is no conflict of interest.

References


