CLASSES OF HARMONIC FUNCTIONS DEFINED BY SALEGEAN-TYPE $q$–DIFFERENTIAL OPERATORS

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Abstract

We consider a complex-valued harmonic functions that are univalent can be written in the form $f = h + \bar{g}$, where $h$ and $g$ are analytic, in a simply connected domain $U$ and sense preserving in $U$, is that $|h'(z)| > |g'(z)|$ in $U$. Making use of Salegean $q$– differential operators, we define a new subclasses harmonic starlike functions and obtain sufficient coefficient bounds, distortion theorems and extreme points for $f$ in the new function class. Moreover, we shown that these necessary coefficient bounds are also sufficient for those functions that have negative coefficients.

2000 Mathematics Subject Classification: Primary 30C45; Secondary 30C50.

Key words: harmonic, univalent, Salegean-type $q$– differential operators.

1 Introduction

A continuous function $f = u + iv$ is a complex-valued harmonic function in a complex domain $\Omega$ if both $u$ and $v$ are real and harmonic in $\Omega$. In any simply connected domain $D \subset \Omega$ we can write $f = h + \bar{g}$ where $h$ and $g$ are analytic in $D$. We call $h$ the analytic part and $g$ the co-analytic part of $f$. A necessary and sufficient condition for $f$ to be locally univalent and sense preserving in $D$ is that $|h'(z)| > |g'(z)|$ in $D$ (see [2]).

Let $\mathcal{H}$ be the family of functions $f = h + \bar{g}$ which are harmonic univalent and sense preserving in the open unit disc $\mathbb{U} = \{ z : |z| < 1 \}$ so that $f$ is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Such functions $f = h + \bar{g} \in \mathcal{H}$ may be expressed by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1. \quad (1)$$

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We note that the family \( \mathcal{H} \) of orientation preserving, normalized harmonic univalent functions reduces to the well known class \( \mathcal{S} \) of normalized univalent functions if the co-analytic part of \( f = h + \overline{g} \) is identically zero, that is \( g \equiv 0 \). We let \( \mathcal{H} \) be the subclass of \( \mathcal{H} \) consisting harmonic functions of the form \( f_m = h + g_m \) where

\[
h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g_m(z) = \frac{(-1)^m}{m} \sum_{n=1}^{\infty} b_n z^n
\]

(2)

so that \( a_n \geq 0 \) and \( b_n \geq 0 \).

We recall the notion of \( q \)-operators or \( q \)-difference operators that play vital roles in the theory of hypergeometric series, quantum physics and operator theory. The application of \( q \)-calculus was initiated by Jackson [4] and Kanas and Răducanu [8] who have used the fractional \( q \)-calculus operators in investigations of certain classes of functions which are analytic in \( U \). For more details on \( q \)-calculus and its applications one can refer to [1, 3, 4, 8] and the references cited therein.

For 0 < \( q < 1 \) the Jackson’s \( q \)-derivative of a function \( f \in \mathcal{S} \) is given as follows [4]

\[
D_q f(z) = \begin{cases} 
\frac{f(z) - f(qz)}{(1-q)z} & \text{for } z \neq 0, \\
 f'(0) & \text{for } z = 0,
\end{cases}
\]

(3)

\[
D_q^2 f(z) = D_q(D_q f(z)).
\]

From (3), we have \( D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1} \) where \( [n]_q = \frac{1-q^n}{1-q} \) is sometimes called the basic number \( n \). If \( q \to 1^- \) then \( [n] \to n \). For \( f \in \mathcal{S} \), Govindaraj and Sivasubramanian [3] considered the Salagean \( q \)-differential operators

\[
D_q^0 f(z) = f(z),
\]

\[
D_q^1 f(z) = zD_q f(z),
\]

\[
D_q^m f(z) = zD_q^m(D_q^{m-1} f(z)),
\]

\[
D_q^m f(z) = z + \sum_{n=2}^{\infty} [n]_q^m a_n z^n \quad (m \in \mathbb{N}_0, z \in U).
\]

We note that if \( \lim_{q \to 1^-} \) then

\[
D_q^m f(z) = z + \sum_{n=2}^{\infty} [n]_q^m a_n z^n \quad (m \in \mathbb{N}_0, z \in U)
\]

is the familiar Salagean derivative[9]. Recently Jahangiri [6] considered a generalized Salagean \( q \)-differential operator for harmonic function \( f = h + \overline{g} \in \mathcal{H} \) defined for \( m > -1 \) by

\[
D_q^m f(z) = D_q^m h(z) + (-1)^m D_q^m g(z) = z + \sum_{n=2}^{\infty} [n]_q^m a_n z^n + (-1)^m \sum_{n=1}^{\infty} [n]_q^m b_n z^n.
\]

(4)
As a generalization of the functions defined in [6], for $0 \leq \alpha < 1$, we let $\mathcal{HR}^m(\lambda, \alpha)$ be the subclass of $\mathcal{H}$ consisting of functions $f = h + \overline{g}$ of the form (1) so that
\[
\Re \left( \frac{D^m_{q+1} f(z)}{(1 - \lambda)D^m_q f(z) + \lambda D^m_{q+1} f(z)} \right) \geq \alpha \tag{5}
\]
where $0 \leq \lambda < 1$, $D^m_q f$ is given by (4) and $z \in \mathbb{U}$. We also let $\mathcal{FR}^m(\lambda, \alpha) = \mathcal{HR}^m(\lambda, \alpha) \cap \mathcal{F}$. Obviously, for $\lambda = 0$ we have $\mathcal{FR}^m(\lambda, \alpha) \equiv \mathcal{FR}^m(\alpha)$ considered in [6]. It is the aim of this paper to obtain sufficient coefficient bounds, distortion theorems and extreme points for functions in $\mathcal{HR}^m(\lambda, \alpha)$. Moreover we show that these necessary coefficient bounds are also sufficient for functions in $\mathcal{FR}^m(\lambda, \alpha)$.

2 Main Results

First we obtain a sufficient coefficient condition for functions in $\mathcal{HR}^m(\lambda, \alpha)$.

**Theorem 1.** Let $f = h + \overline{g}$ be given by (1). If
\[
\sum_{n=1}^{\infty} [n]_q^m \left\{ ([n]_q - \alpha - \alpha \lambda ([n]_q - 1)) |a_n| + ([n]_q + \alpha - \alpha \lambda ([n]_q + 1)) |b_n| \right\} \leq 2(1 - \alpha) \tag{6}
\]
where $a_1 = 1$ and $0 \leq \alpha < 1$, then $f \in \mathcal{HR}^m(\lambda, \alpha)$.

**Proof.** We will show that if (6) holds for the coefficients of $f = h + \overline{g}$ then the required condition (5) is satisfied. We note that (5) can be rewritten as
\[
\Re \left( \frac{D^m_{q+1} h(z) - (-1)^m D^m_{q+1} g(z)}{(1 - \lambda)(D^m_q h(z) + (-1)^m D^m_q g(z)) + \lambda(D^m_{q+1} h(z) - (-1)^m D^m_{q+1} g(z))} \right)
\]
\[
= \Re \left( \frac{A(z)}{B(z)} \right) \geq \alpha
\]
where
\[
A(z) = D^m_{q+1} h(z) - (-1)^m D^m_{q+1} g(z) = z + \sum_{n=2}^{\infty} [n]_q^m a_n z^n - (-1)^m \sum_{n=1}^{\infty} [n]_q^m b_n z^n
\]
and
\[
B(z) = (1 - \lambda)(D^m_q h(z) + (-1)^m D^m_q g(z)) + \lambda(D^m_{q+1} h(z) - (-1)^m D^m_{q+1} g(z))
\]
\[
= z + \sum_{n=2}^{\infty} [n]_q^m (1 - \lambda + \lambda [n]_q) a_n z^n + (-1)^m \sum_{n=1}^{\infty} [n]_q^m (1 - \lambda - \lambda [n]_q) b_n z^n.
\]
Using the fact that $\Re \{ w \} \geq \alpha$ if and only if $|1 - \alpha + w| \geq |1 + \alpha - w|$, it suffices to show that
\[
|A(z) + (1 - \alpha) B(z)| - |A(z) - (1 + \alpha) B(z)| \geq 0. \tag{7}
\]
Substituting for $A(z)$ and $B(z)$ in (7), we get
\[
|A(z) + (1 - \alpha)B(z)| - |A(z) - (1 + \alpha)B(z)|
\]
\[
= |(2 - \alpha)z + \sum_{n=2}^{\infty} [n]_q m \{[n]_q + 1 - \alpha(1 - \lambda + \lambda[n]_q)\} a_n z^n
\]
\[
- \sum_{n=1}^{\infty} |[n]_q - (1 - \alpha)(1 - \lambda - \lambda[n]_q)| b_n |z|^n
\]
\[
- \sum_{n=2}^{\infty} |[n]_q - (1 + \alpha)(1 - \lambda + \lambda[n]_q)| a_n |z|^n
\]
\[
- \sum_{n=1}^{\infty} |[n]_q + (1 + \alpha)(1 - \lambda - \lambda[n]_q)| b_n |z|^n
\]
\[
\geq (2 - \alpha)|z| - \sum_{n=2}^{\infty} [n]_q m \{[n]_q + (1 - \alpha)(1 - \lambda + \lambda[n]_q)\} |a_n| |z|^n
\]
\[
- \sum_{n=1}^{\infty} [n]_q m \{[n]_q - (1 - \alpha)(1 - \lambda - \lambda[n]_q)\} |b_n| |z|^n
\]
\[
- \alpha|z| - \sum_{n=2}^{\infty} [n]_q m \{[n]_q - (1 + \alpha)(1 - \lambda + \lambda[n]_q)\} |a_n| |z|^n
\]
\[
- \sum_{n=1}^{\infty} [n]_q m \{[n]_q + (1 + \alpha)(1 - \lambda - \lambda[n]_q)\} |b_n| |z|^n
\]
\[
\geq 2(1 - \alpha)|z| \left( 2 - \sum_{n=1}^{\infty} [n]_q m \left[ \frac{[n]_q - \alpha - \alpha \lambda([n]_q - 1)}{1 - \alpha} \right] |a_n| + \frac{[n]_q + \alpha - \alpha \lambda([n]_q + 1)}{1 - \alpha} |b_n| \right) |z|^{n-1}
\]
\[
\geq 2(1 - \alpha) \left( 2 - \sum_{n=1}^{\infty} [n]_q m \left[ \frac{[n]_q - \alpha - \alpha \lambda([n]_q - 1)}{1 - \alpha} \right] |a_n| + \frac{[n]_q + \alpha - \alpha \lambda([n]_q + 1)}{1 - \alpha} |b_n| \right)
\]

The above expression is non negative by (6) and so $f(z) \in \mathcal{H}^m_\phi(\lambda, \alpha)$. \hfill \Box

For $\lambda = 0$ we obtain the following corollary which is also given by Jahangiri [6].

**Corollary 1.** Let $f = h + \overline{g}$ be given by (1). If
\[
\sum_{n=1}^{\infty} [n]_q m \{([n]_q - \alpha)|a_n| + ([n]_q + \alpha)|b_n|\} \leq 2(1 - \alpha)
\]
where $a_1 = 1$ and $0 \leq \alpha < 1$, then $f \in \mathcal{H}^m_\phi(\alpha)$.  

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The starlikeness of the functions given in Theorem 1 follows from Theorem 1 given in [5] and noticing that
\[ [n]_q - \alpha - \alpha \lambda ([n]_q - 1) \leq [n]_q - \alpha \leq n - \alpha \]
and
\[ [n]_q + \alpha - \alpha \lambda ([n]_q + 1) \leq [n]_q + \alpha \leq n + \alpha. \]

Next we show that the coefficient bounds (6) are also sufficient for functions in \( \mathcal{FR}_q^m(\lambda, \alpha) \).

**Theorem 2.** Let \( f_m = h + \overline{g}_m \) given by (2) is in \( \mathcal{FR}_q^m(\lambda, \alpha) \) if and only if
\[
\sum_{n=1}^{\infty} [n]_q^m \{([n]_q - \alpha - \alpha \lambda ([n]_q - 1)) a_n + ([n]_q + \alpha - \alpha \lambda ([n]_q + 1)) b_n \} \leq 2(1 - \alpha)
\]
where \( a_1 = 1 \) and \( 0 \leq \alpha < 1 \).

**Proof.** Since \( \mathcal{FR}_q^m(\lambda, \alpha) \subset \mathcal{FR}_q(\lambda, \alpha) \), we only need to prove the "only if" part of the theorem. To this end, for functions \( f_m = h + \overline{g}_m \) in \( \mathcal{FR}_q^m(\lambda, \alpha) \) we must have
\[
\Re \left( \frac{D_q^{m+1} f_m(z)}{(1 - \lambda) D_q^m f_m(z) + \lambda D_q^{m+1} f_m(z)} \right) \geq \alpha
\]
or equivalently,
\[
\Re \left( \frac{(1 - \alpha) z - \sum_{n=2}^{\infty} [n]_q^m \{([n]_q - \alpha - \alpha \lambda ([n]_q - 1)) a_n z^n \}}{z - \sum_{n=2}^{\infty} [n]_q^m (1 - \lambda + \lambda [n]_q) a_n z^n + (-1)^{2m} \sum_{n=1}^{\infty} [n]_q^m (1 - \lambda - \lambda [n]_q) b_n \overline{z}^n} \right)
\]
and
\[
\Re \left( \frac{(-1)^{2m} \sum_{n=1}^{\infty} [n]_q^m \{([n]_q + \alpha - \alpha \lambda ([n]_q + 1)) b_n \overline{z}^n \}}{z - \sum_{n=2}^{\infty} [n]_q^m (1 - \lambda + \lambda [n]_q) a_n z^n + (-1)^{2m} \sum_{n=1}^{\infty} [n]_q^m (1 - \lambda - \lambda [n]_q) b_n \overline{z}^n} \right) \geq 0.
\]

The above condition must hold for all values of \( z \) in \( \mathbb{U} \). Upon choosing the values of \( z \) on the positive real axis where \( 0 \leq z = r < 1 \), we must have
\[
\left( (1 - \alpha) - \sum_{n=2}^{\infty} [n]_q^m \{([n]_q - \alpha - \alpha \lambda ([n]_q - 1)) a_n r^{n-1} \} \right)
\]
\[
- \sum_{n=1}^{\infty} [n]_q^m \{([n]_q + \alpha - \alpha \lambda ([n]_q + 1)) b_n r^{n-1} \} \times
\]
\[
\left( 1 - \sum_{n=2}^{\infty} [n]_q^m (1 - \lambda + [n]_q \lambda) a_n r^{n-1} + \sum_{n=1}^{\infty} [n]_q^m (1 - \lambda - [n]_q \lambda) b_n r^{n-1} \right)^{-1} \geq 0.
\]
If the condition (8) does not hold, then the numerator in the above inequality is negative for \( r \) sufficiently close to 1. Hence, there exists \( z_0 = r_0 \) in \((0,1)\) for which the left hand side of the above inequality is negative. This contradicts the required condition for \( f(z) \in \mathcal{F}R_m^q(\lambda, \alpha) \) and so the proof is complete.

Next we determine the extreme points of closed convex hulls of \( \mathcal{F}R_m^q(\lambda, \alpha) \) denoted by \( \text{clco}\mathcal{F}R_m^q(\lambda, \alpha) \).

**Theorem 3.** A function \( f_m(z) \in \mathcal{F}R_m^q(\lambda, \alpha) \) if and only if

\[
f_m(z) = \sum_{n=1}^{\infty} \left( X_n h_n(z) + Y_n g_{nm}(z) \right)
\]

where \( h_1(z) = z, \ h_n(z) = z - \frac{1-\alpha}{[n]_q ([n]_q - \alpha - \alpha \lambda ([n]_q - 1))} z^n; \ (n \geq 2), \) and \( g_{nm}(z) = z + \frac{(-1)^m(1-\alpha)}{[n]_q ([n]_q + \alpha - \alpha \lambda ([n]_q + 1))} z^n; \ (n \geq 2), \sum_{n=2}^{\infty} (X_n + Y_n) = 1, \ X_n \geq 0 \text{ and } Y_n \geq 0.

In particular, the extreme points of \( \mathcal{F}R_m^q(\lambda, \alpha) \) are \( \{h_n\} \) and \( \{g_{nm}\} \).

**Proof.** First, we note that for \( f_m \) as given in the theorem, we may write

\[
f_m(z) = \sum_{n=1}^{\infty} \left( X_n h_n(z) + Y_n g_{nm}(z) \right)
\]

\[
= \sum_{n=1}^{\infty} (X_n + Y_n) z - \sum_{n=2}^{\infty} \frac{1-\alpha}{[n]_q ([n]_q - \alpha - \alpha \lambda ([n]_q - 1))} X_n z^n
\]

\[
+ (-1)^m \sum_{n=1}^{\infty} \frac{1-\alpha}{[n]_q ([n]_q + \alpha - \alpha \lambda ([n]_q + 1))} Y_n z^n
\]

\[
= z - \sum_{n=2}^{\infty} A_n z^n + (-1)^m \sum_{n=1}^{\infty} B_n z^n,
\]

where

\[
A_n = \frac{1-\alpha}{[n]_q ([n]_q - \alpha - \alpha \lambda ([n]_q - 1))} X_n,
\]

\[
B_n = \frac{1-\alpha}{[n]_q ([n]_q + \alpha - \alpha \lambda ([n]_q + 1))} Y_n.
\]

Therefore

\[
\sum_{n=2}^{\infty} \frac{[n]_q ([n]_q - \alpha - \alpha \lambda ([n]_q - 1))}{1-\alpha} A_n + \sum_{n=1}^{\infty} \frac{[n]_q ([n]_q + \alpha - \alpha \lambda ([n]_q + 1))}{1-\alpha} B_n
\]

\[
= \sum_{n=2}^{\infty} X_n + \sum_{n=1}^{\infty} Y_n = 1 - X_1 \leq 1,
\]

\[
= 1.
\]
and hence \( f_m(z) \in cl\mathcal{FR}_q^m(\lambda, \alpha) \). Conversely, suppose \( f_m(z) \in cl\mathcal{FR}_q^m(\lambda, \alpha) \).

Set \( X_n = \frac{[n]_q^m([n]_q - \alpha - \alpha\lambda([n]_q - 1))}{1 - \alpha} A_n \) and \( Y_n = \frac{[n]_q^m([n]_q + \alpha - \alpha\lambda([n]_q - 1))}{1 - \alpha} B_n \), where \( \sum_{n=1}^{\infty} (X_n + Y_n) = 1 \). Then

\[
\begin{align*}
X_n &= \frac{1}{1 - \alpha} \frac{1}{[n]_q}[n]_q \{|n|_q - \alpha - \alpha\lambda([n]_q - 1)\} X_n z^n \\
Y_n &= \frac{1}{1 - \alpha} \frac{1}{[n]_q}[n]_q \{|n|_q + \alpha - \alpha\lambda([n]_q - 1)\} Y_n z^n
\end{align*}
\]

\( f_m(z) = z - \sum_{n=2}^{\infty} a_n z^n + (-1)^m \sum_{n=1}^{\infty} b_n z^n \)

\[
= z - \sum_{n=2}^{\infty} \frac{1}{[n]_q^m([n]_q - \alpha - \alpha\lambda([n]_q - 1))} X_n z^n + (-1)^m \sum_{n=1}^{\infty} \frac{1}{[n]_q^m([n]_q + \alpha - \alpha\lambda([n]_q - 1))} Y_n z^n
\]

\[
= z - \sum_{n=2}^{\infty} (h_n(z) - z) X_n + \sum_{n=1}^{\infty} (g_n(z) - z) Y_n
\]

\[
= \sum_{n=1}^{\infty} (X_n h_n(z) + Y_n g_n(z))
\]

as required. \( \square \)

Next we give distortion bounds and a covering result for the class \( \mathcal{FR}_q^m(\lambda, \alpha) \).

**Theorem 4.** Let \( f_m \in \mathcal{FR}_q^m(\lambda, \alpha) \). Then for \( |z| = r < 1 \), we have

\[
(1 - b_1) r - \frac{1}{|2|_q^m} \left( \frac{1 - \alpha}{|2|_q - \alpha - \alpha\lambda} - \frac{1 + \alpha}{|2|_q - \alpha - \alpha\lambda} b_1 \right) r^2 \leq |f_m(z)|
\]

\[
\leq (1 + b_1) r + \frac{1}{|2|_q^m} \left( \frac{1 - \alpha}{|2|_q - \alpha - \alpha\lambda} - \frac{1 + \alpha}{|2|_q - \alpha - \alpha\lambda} b_1 \right) r^2.
\]

**Proof.** We only prove the right hand inequality. Taking the absolute value of \( f_m(z) \), we obtain

\[
|f_m(z)| = \left| z + \sum_{n=2}^{\infty} a_n z^n + (-1)^m \sum_{n=1}^{\infty} b_n z^n \right|
\]

\[
\leq (1 + b_1)|z| + \sum_{n=2}^{\infty} (a_n + b_n)|z|^n
\]

\[
\leq (1 + b_1) r + \sum_{n=2}^{\infty} (a_n + b_n)r^2
\]
The proof of the left hand inequality is similar and is omitted.

Corollary 2. Let $f_m(z) \in \overline{F\mathcal{R}}^m_q(\lambda, \alpha)$. Then

$$\left\{ w : |w| < \frac{[2]^m_q([2]_q - \alpha - \alpha \lambda) - 1 + \alpha}{[2]^m_q([2]_q - \alpha - \alpha \lambda)} - \frac{[2]^m_q([2]_q - \alpha - \alpha \lambda) - (1 + \alpha) b_1}{[2]^m_q([2]_q - \alpha - \alpha \lambda)} \right\} \subset f_m(U).$$

Proof. For completeness, we provide a brief justification. Using the left hand inequality of Theorem 4 and letting $r \to 1$, it follows that

$$(1 - b_1) - \frac{1}{[2]^m_q([2]_q - \alpha - \alpha \lambda)} \left( \frac{1 - \alpha}{[2]_q - \alpha - \alpha \lambda} - \frac{1}{1 - \alpha} b_1 \right) = (1 - b_1) - \frac{1}{[2]^m_q([2]_q - \alpha - \alpha \lambda)} \left[ 1 - \alpha - (1 + \alpha) b_1 \right] = (1 - b_1) [2]^m_q([2]_q - \alpha - \alpha \lambda) - (1 - \alpha) + (1 + \alpha) b_1$$

$$= \frac{[2]^m_q([2]_q - \alpha - \alpha \lambda) - [2]^m_q([2]_q - \alpha - \alpha \lambda) b_1 - (1 - \alpha) + (1 + \alpha) b_1}{[2]^m_q([2]_q - \alpha - \alpha \lambda)}$$

$$= \frac{[2]^m_q([2]_q - \alpha - \alpha \lambda) - 1 + \alpha - [2]^m_q([2]_q - \alpha - \alpha \lambda) - (1 + \alpha) b_1}{[2]^m_q([2]_q - \alpha - \alpha \lambda)}$$

$$= \frac{[2]^m_q([2]_q - \alpha - \alpha \lambda) - 1 + \alpha}{[2]^m_q([2]_q - \alpha - \alpha \lambda)} - \frac{[2]^m_q([2]_q - \alpha - \alpha \lambda) - (1 + \alpha) b_1}{[2]^m_q([2]_q - \alpha - \alpha \lambda)} \subset f_m(U).$$

Finally we show that class $\overline{F\mathcal{R}}^m_q(\lambda, \alpha)$ is closed under convex combinations.

Theorem 5. The family $\overline{F\mathcal{R}}^m_q(\lambda, \alpha)$ is closed under convex combinations.

Proof. For $i = 1, 2, \ldots$, suppose that $f_{m_i} \in \overline{F\mathcal{R}}^m_q(\lambda, \alpha)$ where

$$f_{m_i}(z) = z - \sum_{n=2}^{\infty} a_{i,n} z^n + (-1)^m \sum_{n=2}^{\infty} b_{i,n} z^n.$$
Then, by Theorem 2
\[ \sum_{n=2}^{\infty} \frac{[n]_q^m ([n]_q - \alpha - \alpha \lambda ([n]_q - 1))}{1 - \alpha} a_{i,n} + \sum_{n=1}^{\infty} \frac{[n]_q^m ([n]_q + \alpha - \alpha \lambda ([n]_q + 1))}{1 - \alpha} b_{i,n} \leq 1. \]

For \( \sum_{i=1}^{\infty} t_i, 0 \leq t_i \leq 1, \) the convex combination of \( f_i \) may be written as
\[ \sum_{i=1}^{\infty} t_i f_{m_i}(z) = z - \sum_{n=2}^{\infty} \left( \sum_{i=1}^{\infty} t_i a_{i,n} \right) z^n + (-1)^m \sum_{n=1}^{\infty} \left( \sum_{i=1}^{\infty} t_i b_{i,n} \right) z^n. \]

Using the inequality (8), we obtain
\[ \sum_{n=2}^{\infty} \frac{[n]_q^m ([n]_q - \alpha - \alpha \lambda ([n]_q - 1))}{1 - \alpha} \left( \sum_{i=1}^{\infty} t_i a_{i,n} \right) + \sum_{n=1}^{\infty} \frac{[n]_q^m ([n]_q + \alpha - \alpha \lambda ([n]_q + 1))}{1 - \alpha} \left( \sum_{i=1}^{\infty} t_i b_{i,n} \right) \]
\[ = \sum_{i=1}^{\infty} t_i \left( \sum_{n=2}^{\infty} \frac{[n]_q^m ([n]_q - \alpha - \alpha \lambda ([n]_q - 1))}{1 - \alpha} a_{i,n} \right. \]
\[ + \left. \sum_{n=1}^{\infty} \frac{[n]_q^m ([n]_q + \alpha - \alpha \lambda ([n]_q + 1))}{1 - \alpha} b_{i,n} \right) \]
\[ \leq \sum_{i=1}^{\infty} t_i = 1, \]
and therefore \( \sum_{i=1}^{\infty} t_i f_{m_i} \in \overline{HR}^m_q(\lambda, \alpha). \)

**Concluding Remarks:** The results of this paper for the special case \( \lambda = 0 \) yield analogous results obtained in [6]. Furthermore, by letting \( \lim_{q \to 1} \) and taking \( \lambda = 0 \) and \( m = 0 \) we obtain the analogous results for the classes studied in [7] and [5], respectively. Moreover, if we let \( \alpha = 0 \) we obtain the results given in [10].

**Acknowledgement:** We thank the referees for their valuable suggestions.

**References**


