SOME APPLICATIONS OF CERTAIN NEW TYPES OF SETS
IN GTS VIA HEREDITARY CLASSES

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Abstract

In this paper we introduce certain new types of sets in a generalized topological space via hereditary classes and investigate their several properties. In the process we achieve some nice applications of these newly defined sets to study a few lower separation properties viz $\mu^*-R_0$, $\mu^*-R_1$ and $\mu^*-T_{1/2}$ spaces.

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1 Introduction

The idea of generalized topology [2] was introduced by A. Császár in 2002 and since then there has been a growing trend to study this concept in different perspectives. In 2007, A. Császár [5] introduced the notion of hereditary class in generalized topological space and subsequently many papers (e.g. see [7, 8, 10, 11, 12, 13, 15]) appeared in the recent literature. In this article, there is another attempt to introduce and investigate some new kind of sets in a generalized topological space with a hereditary class. Also, we give some applications of these sets by characterizing certain separation axioms viz. $\mu^*-R_0$, $\mu^*-R_1$ and $\mu^*-T_{1/2}$.

A collection $\mu$ of subsets of a set $X$ is called a generalized topology [2] on $X$ if $\phi \in \mu$ and $\mu$ is closed under arbitrary union; the pair $(X, \mu)$ is called a generalized topological space (GTS, in short). The members of $\mu$ are called $\mu$-open sets and their complements are called $\mu$-closed sets in $(X, \mu)$. According to [1], for $A \subseteq X$, the union of all $\mu$-open subsets of $X$, each contained in $A$ is called $\mu$-interior of $A$ and is denoted by $i_{\mu}(A)$; the map $i_{\mu} : \exp X \to \exp X$ is monotone (i.e., $A \subseteq B \Rightarrow i_{\mu}(A) \subseteq i_{\mu}(B)$), restricting (i.e., $i_{\mu}(A) \subseteq A$ for $A \subseteq X$) and

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idempotent (i.e., \( i_\mu(i_\mu(A)) = i_\mu(A) \)), where \( \exp X \) denotes the set of all subsets of \( X \). The generalized closure of a subset \( A \) of \( X \), denoted by \( c_\mu(A) \), is the intersection of all \( \mu \)-closed subsets of \( X \) each containing \( A \); the map \( c_\mu : \exp X \to \exp X \) is monotone, idempotent and enlarging (i.e., \( A \subseteq c_\mu(A) \) for \( A \subseteq X \)). Moreover \( c_\mu(X \setminus A) = X \setminus i_\mu(A) \) \([4]\). 

A family \( \mathcal{H} \) of subsets of \( X \) is said to be a hereditary class \([5]\) on \( X \) if \( A \in \mathcal{H} \) and \( B \subseteq A \) implies \( B \in \mathcal{H} \). For a GTS \((X, \mu)\) with a hereditary class \( \mathcal{H} \), a subset \( A^*(\mu, \mathcal{H}) \) or simply \( A^* \) of \( X \) is defined by \( A^* = \{ x \in X : U \cap A \notin \mathcal{H} \text{ for every } U \in \mu \text{ containing } x \} \) \([5]\), for each \( A \subseteq X \). In \([5]\), it was also shown that for \( A \subseteq X \) if \( c_\mu^*(A) = A \cup A^* \), then \( \mu^*(\mu, \mathcal{H}) \) (or simply \( \mu^* \)) = \( \{ A \subseteq X : c_\mu^*(X \setminus A) = X \setminus A \} \) is a generalized topology on \( X \) with \( \mu \subseteq \mu^* \). Moreover, the map \( c_\mu^* \) is monotone, enlarging and idempotent. The elements of \( \mu^* \) are called \( \mu^* \)-open sets. The complements of \( \mu^* \)-open sets are called \( \mu^* \)-closed sets and equivalently \( A \) is a \( \mu^* \)-closed set iff \( A^* \subseteq A \) \([5]\). 

In Section 2 of this paper, we introduce two types of sets viz. \( \wedge_\mu^* \)-set and \( \vee_\mu^* \)-set in a GTS with a hereditary class and study some of their properties. In \([9]\), we investigated \( \mu^*-R_0 \), \( \mu^*-R_1 \) and \( \mu^*-T_\frac{2}{2} \) spaces in a GTS with a hereditary class. In the last section of this article, we investigate some lower separation axioms viz. \( \mu^*-R_0 \), \( \mu^*-R_1 \) and \( \mu^*-T_\frac{2}{2} \) with the help of different types of sets introduced in Sections 2 and 3.

**Definition 1.** \([6]\) Let \((X, \mu)\) be a GTS and \( A \subseteq X \). The subsets \( \wedge_\mu(A) \) and \( \vee_\mu(A) \) are defined by 

\[
\wedge_\mu(A) = \begin{cases} 
\cap \{ U : A \subseteq U, U \text{ is } \mu \text{-open sets} \}, & \text{if } \exists U \in \mu \text{ such that } A \subseteq U; \\
X, & \text{otherwise}
\end{cases}
\]

\[
\vee_\mu(A) = \begin{cases} 
\cup \{ F : F \subseteq A, F \text{ is } \mu \text{-closed} \}, & \text{if } \exists \mu \text{-closed } F \text{ such that } F \subseteq A; \\
\phi, & \text{otherwise}
\end{cases}
\]

**Definition 2.** \([6]\) A subset \( A \) of a GTS \((X, \mu)\) is called a \( \wedge_\mu \)-set (\( \vee_\mu \)-set) if \( A = \wedge_\mu(A) \) (respectively, if \( A = \vee_\mu(A) \)). 

**Theorem 1.** \([6]\) Let \((X, \mu)\) be a GTS and \( A \) be any subset of \( X \). Then \( \wedge_\mu(A) = \{ x \in X : c_\mu(\{x\}) \cap A \neq \phi \} \).

2 \( \wedge_\mu^* \) and \( \vee_\mu^* \)-sets 

The intent of this section is to introduce two types of sets viz. \( \wedge_\mu^* \)-sets and \( \vee_\mu^* \)-sets, and characterize \( \mu^* \)-g-closed sets with the help of these types of sets. Before we begin this section, we observe that \( \mathcal{M}_\mu = \cup \{ M \mid M \in \mu \} \) is the largest \( \mu \)-open set of \( X \), and certainly if \( B \) is a \( \mu \)-closed set then \( X \setminus \mathcal{M}_\mu \subseteq B \subseteq X \).

**Definition 3.** Let \((X, \mu)\) be a GTS with a hereditary class \( \mathcal{H} \) and \( A \subseteq X \). We define
Let $X$ be a GTS with a hereditary class $\mathcal{H}$. Then \( \bigwedge^\ast_{\mu}(A) = \{ x \in X : c^\ast_{\mu}(\{x\}) \cap A \neq \emptyset \} \) for each $A \subseteq X$.

Proof. (i) $A \subseteq \bigwedge^\ast_{\mu}(A)$.

(ii) If $A$ is $\mu^\ast$-open, then $A = \bigwedge^\ast_{\mu}(A)$.

(iii) If $A \subseteq B$, then $\bigwedge^\ast_{\mu}(A) \subseteq \bigwedge^\ast_{\mu}(B)$.

(iv) $\bigwedge^\ast_{\mu}(\bigwedge^\ast_{\mu}(B)) = \bigwedge^\ast_{\mu}(A)$.

(v) $\bigwedge^\ast_{\mu}(\cap\{A_\alpha : \alpha \in \Delta\}) \subseteq \cap\{\bigwedge^\ast_{\mu}(A_\alpha) : \alpha \in \Delta\}$.

(vi) $\bigwedge^\ast_{\mu}(\cup\{A_\alpha : \alpha \in \Delta\}) = \cup\{\bigwedge^\ast_{\mu}(A_\alpha) : \alpha \in \Delta\}$.

Proof. (i) and (ii) follow from the definition.

(iii) Let $A \subseteq B$. If $x \notin \bigwedge^\ast_{\mu}(B)$, then there exists a $\mu^\ast$-open set $U$ such that $B \subseteq U$ and $x \notin U$. Since $A \subseteq B \subseteq U$, then from the definition of $\bigwedge^\ast_{\mu}(A)$, we have $x \notin \bigwedge^\ast_{\mu}(A)$ and hence $\bigwedge^\ast_{\mu}(A) \subseteq \bigwedge^\ast_{\mu}(B)$.

(iv) By (i), we have $\bigwedge^\ast_{\mu}(\bigwedge^\ast_{\mu}(A)) \supseteq \bigwedge^\ast_{\mu}(A)$. Suppose that $x \notin \bigwedge^\ast_{\mu}(A)$. Then there exists a $\mu^\ast$-open set $U$ such that $A \subseteq U$ and $x \notin U$. Since $A \subseteq \bigwedge^\ast_{\mu}(A) \subseteq U$, from the definition of $\bigwedge^\ast_{\mu}(\bigwedge^\ast_{\mu}(A))$, we have $\bigwedge^\ast_{\mu}(\bigwedge^\ast_{\mu}(A)) \subseteq U$, and hence $x \notin \bigwedge^\ast_{\mu}(\bigwedge^\ast_{\mu}(A))$, that is $\bigwedge^\ast_{\mu}(\bigwedge^\ast_{\mu}(A)) \subseteq \bigwedge^\ast_{\mu}(A)$. Thus $\bigwedge^\ast_{\mu}(\bigwedge^\ast_{\mu}(A)) = \bigwedge^\ast_{\mu}(A)$.

(v) It follows from (ii) and (iii).

(vi) We have $\bigwedge^\ast_{\mu}(A_\alpha) \subseteq \bigwedge^\ast_{\mu}(\bigcup_{\alpha \in \Delta} A_\alpha)$ and hence $\bigcup_{\alpha \in \Delta} \bigwedge^\ast_{\mu}(A_\alpha) \subseteq \bigwedge^\ast_{\mu}(\bigcup_{\alpha \in \Delta} A_\alpha)$. Next, let $x \notin \bigcup_{\alpha \in \Delta} \bigwedge^\ast_{\mu}(A_\alpha)$. Then $x \notin \bigwedge^\ast_{\mu}(A_\alpha)$ for each $\alpha \in \Delta$ and so there exists a $\mu^\ast$-open set $U_\alpha$ such that $A_\alpha \subseteq U_\alpha$ and $x \notin U_\alpha$. Let $U = \bigcup U_\alpha$. Then $U \in \mu^\ast$ such that $\cup A_\alpha \subseteq U$ and $x \notin U$, and hence $x \notin \bigwedge^\ast_{\mu}(\cup A_\alpha)$.

Remark 1. In (v) of Theorem 3, the equality does not hold in general, even if $\Delta$ is a finite index set. See the following example.

Example 1. Let $X = \{a,b,c,d\}$. Consider a GT $\mu$ on $X$, where $\mu = \{\emptyset, \{a\}, \{d\}, \{a,d\}, \{a,c\}, \{a,c,d\}\}$ and a hereditary class $\mathcal{H} = \{\emptyset, \{b\}, \{c\}\}$. Then $\mu^\ast = \{\emptyset, \{a\}, \{d\}, \{a,d\}, \{a,c\}, \{a,c,d\}\}$. Let us consider $A = \{a,d\}$ and $B = \{c,d\}$. Then $\bigwedge^\ast_{\mu}(A) = \{a,d\}$, $\bigwedge^\ast_{\mu}(B) = \{a,c,d\}$, and $\bigwedge^\ast_{\mu}(A \cap B) = \{d\}$. Thus $\bigwedge^\ast_{\mu}(A \cap B) \neq \bigwedge^\ast_{\mu}(A) \cap \bigwedge^\ast_{\mu}(B)$.
Lemma 1. Let $(X, \mu)$ be a GTS with a hereditary class $\mathcal{K}$. Then $\bigwedge^*_\mu(X \setminus A) = X \setminus \bigvee^*_\mu(A)$ for every $A \subseteq X$.

**Proof.** We have $X \setminus \bigvee^*_\mu(A) = X \setminus (\bigcup \{F : F \subseteq A \text{ and } F \text{ is a } \mu^*-\text{closed set}\}) = \bigcap \{X \setminus A \subseteq X \setminus F \text{ and } X \setminus F \text{ is a } \mu^*-\text{open set} \} = \bigwedge^*_\mu(X \setminus A)$.

Using the above lemma and Theorem 3, we have the following result:

**Theorem 4.** For subsets $A, B, A_\alpha (\alpha \in \Delta)$ of a GTS $(X, \mu)$ with a hereditary class $\mathcal{K}$, the following properties hold:

(i) $\bigvee^*_\mu(A) \subseteq A$.

(ii) If $A$ is $\mu^*$-closed, then $A = \bigvee^*_\mu(A)$.

(iii) If $A \subseteq B$, then $\bigvee^*_\mu(A) \subseteq \bigvee^*_\mu(B)$.

(iv) $\bigvee^*_\mu(\bigvee^*_\mu(A)) = \bigvee^*_\mu(A)$.

(v) $\bigvee^*_\mu(\bigcup \{A_\alpha : \alpha \in \Delta\}) = \bigcup \{\bigvee^*_\mu(A_\alpha) : \alpha \in \Delta\}$.

(vi) $\bigcup \{\bigvee^*_\mu(A_\alpha) : \alpha \in \Delta\} \subseteq \bigvee^*_\mu(\bigcup \{A_\alpha : \alpha \in \Delta\})$.

**Definition 4.** Let $(X, \mu)$ be a GTS with a hereditary class $\mathcal{K}$. A subset $A$ of $X$ is said to be a

(i) $\bigwedge^*_\mu$-set if $A = \bigwedge^*_\mu(A)$,

(ii) $\bigvee^*_\mu$-set if $A = \bigvee^*_\mu(A)$.

Therefore a subset $A$ of $X$ is a $\bigwedge^*_\mu$-set if and only if $X \setminus A$ is a $\bigvee^*_\mu$-set.

**Theorem 5.** Let $(X, \mu)$ be a GTS with a hereditary class $\mathcal{K}$. Then the following statements hold:

(a) $\phi$ is a $\bigwedge^*_\mu$-set and $X$ is a $\bigvee^*_\mu$-set.

(b) Arbitrary union of $\bigwedge^*_\mu$-sets is a $\bigwedge^*_\mu$-set.

(c) Arbitrary intersection of $\bigvee^*_\mu$-sets is a $\bigvee^*_\mu$-set.

**Proof.** (a) Clear.

(b) Let $\{A_\alpha : \alpha \in \Delta\}$ be an arbitrary family of $\bigwedge^*_\mu$-sets. Then $A_\alpha = \bigwedge^*_\mu(A_\alpha)$, for each $\alpha \in \Delta$. Let $A = \bigcup \{A_\alpha : \alpha \in \Delta\}$. Then by (vi) of Theorem 3, we have $\bigwedge^*_\mu(A) = A$ and hence $A$ is a $\bigwedge^*_\mu$-set.

(c) It follows from Lemma 1 and (b) above.

**Definition 5.** [9] A subset $A$ of a GTS $(X, \mu)$ with a hereditary class $\mathcal{K}$ is said to be $\mu^*$-g-closed if $c_\mu(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\mu^*$-open. The complement of a $\mu^*$-g-closed set is $\mu^*$-g-open.

**Theorem 6.** Let $(X, \mu)$ be a GTS with a hereditary class $\mathcal{K}$ and $A \subseteq X$. Then $A$ is $\mu^*$-g-closed if and only if $c_\mu(A) \subseteq \bigwedge^*_\mu(A)$.

**Proof.** Let $A$ be a $\mu^*$-g-closed set and $x \in c_\mu(A)$. If $x \notin \bigwedge^*_\mu(A)$, then there exists a $\mu^*$-open set $U$ containing $A$ such that $x \notin U$. Now since $A$ is $\mu^*$-g-closed and $A \subseteq U$, where $U$ is $\mu^*$-open, it follows that $c_\mu(A) \subseteq U$ and thus $x \notin c_\mu(A)$, a contradiction. Therefore $c_\mu(A) \subseteq \bigwedge^*_\mu(A)$.

Conversely suppose that $c_\mu(A) \subseteq \bigwedge^*_\mu(A)$. Let $A \subseteq U$, where $U$ is $\mu^*$-open. Then $\bigwedge^*_\mu(A) \subseteq U$ and hence $c_\mu(A) \subseteq U$. Therefore $A$ is $\mu^*$-g-closed.
Corollary 1. Let \((X, \mu)\) be a GTS with a hereditary class \(\mathcal{H}\) and \(A \subseteq X\). Then \(A\) is \(\mu^*\)-g-open if and only if \(\bigwedge_\mu^*(A) \subseteq i_\mu(A)\).

Corollary 2. Let \((X, \mu)\) be a GTS with a hereditary class \(\mathcal{H}\) and \(A\) be a \(\bigwedge_\mu^*\)-set. Then \(A\) is \(\mu^*\)-g-closed if and only if \(A\) is \(\mu\)-closed in \((X, \mu)\).

Proof. Suppose that \(A\) is \(\mu^*\)-g-closed. Then by using Theorem 6, we have \(c_\mu(A) \subseteq \bigwedge_\mu^*(A) = A\). Thus \(A\) is \(\mu\)-closed.

The converse is obvious. \(\square\)

Corollary 3. Let \((X, \mu)\) be a GTS with a hereditary class \(\mathcal{H}\) and \(A\) be a \(\bigvee_\mu^*\)-set. Then \(A\) is \(\mu^*\)-g-open if and only if \(A\) is \(\mu\)-open in \((X, \mu)\).

Theorem 7. Let \((X, \mu)\) be a GTS with a hereditary class \(\mathcal{H}\) and \(A \subseteq X\). Then \(A\) is \(\mu^*\)-g-closed if \(\bigwedge_\mu^*(A)\) is \(\mu^*\)-g-closed.

Proof. Let \(\bigwedge_\mu^*(A)\) be \(\mu^*\)-g-closed. Suppose that \(A \subseteq U\), where \(U\) is \(\mu^*\)-open. Then \(\bigwedge_\mu^*(A) \subseteq U\). Since \(\bigwedge_\mu^*(A)\) is \(\mu^*\)-g-closed, it follows that \(c_\mu(\bigwedge_\mu^*(A)) \subseteq U\).

Since \(A \subseteq \bigwedge_\mu^*(A) \subseteq U\), we have \(c_\mu(A) \subseteq c_\mu(\bigwedge_\mu^*(A)) \subseteq U\). i.e., \(c_\mu(A) \subseteq U\) and thus \(A\) is a \(\mu^*\)-g-closed set. \(\square\)

Remark 2. The converse of the above theorem is false as shown in the following example.

Example 2. Consider a GT \(\mu\) and a hereditary class \(\mathcal{H}\) on \(X = \{a, b, c\}\), where \(\mu = \{\phi, \{a, b\}, \{b, c\}, X\}\) and \(\mathcal{H} = \{\phi, \{a\}, \{c\}\}\). Then \(A = \{a\}\) is a \(\mu^*\)-g-closed but \(\bigwedge_\mu^*(A) = \{a, b\}\) which is not a \(\mu^*\)-g-closed set, since \(c_\mu(\{a, b\}) = X \not\subseteq \bigwedge_\mu^*(\{a, b\}) = \{a, b\}\) (refer to Theorem 6).

3 Generalized \(\bigwedge_\mu^*\) and \(\bigvee_\mu^*\)-sets

In this section, we introduce and study two other types of sets viz. \(g, \bigwedge_\mu^*\)-sets, \(g, \bigvee_\mu^*\)-sets. We discuss several properties of these sets, a few of which involve sets introduced in the previous section. We start this section by recalling the following definition from [16]:

Definition 6. A subset \(A\) of a GTS \((X, \mu)\) is said to be a generalized \(\bigwedge_\mu\)-set (\(g, \bigwedge_\mu\)-set, in short) if \(\bigwedge_\mu(A) \subseteq F\), whenever \(A \subseteq F\) and \(F\) is \(\mu\)-closed in \(X\).

A subset \(A\) of \(X\) is said to be a \(g, \bigvee_\mu\)-set if \(X \setminus A\) is a \(g, \bigwedge_\mu\)-set.

In an analogous way we define generalized \(\bigwedge_\mu\)-sets in our setting as follows:

Definition 7. Let \((X, \mu)\) be a GTS with a hereditary class \(\mathcal{H}\). A subset \(A\) of \(X\) is said to be a generalized \(\bigwedge_\mu^*\)-set (\(g, \bigwedge_\mu^*\)-set, in short) if \(\bigwedge_\mu^*(A) \subseteq F\), whenever \(F\) is \(\mu\)-closed in \(X\) and \(A \subseteq F\).

A subset \(A\) of \(X\) is said to be a \(g, \bigvee_\mu^*\)-set if \(X \setminus A\) is a \(g, \bigwedge_\mu^*\)-set.
Remark 3. (i) Every $g.\land^*_\mu$-set ($g.\lor^*_\mu$-set) is a $g.\land^*_\mu$-set (resp. $g.\lor^*_\mu$-set). But the converse is false (see Example 3(a)).
(ii) Every $\land^*_\mu$-set ($\lor^*_\mu$-set) is a $g.\land^*_\mu$-set (resp. $g.\lor^*_\mu$-set). But the converse is false (see Example 3(b)).

Example 3. (a) Let $X = \{a, b, c\}$, $\mu = \{\phi, \{c\}, \{a, b\}, \{b, c\}, X\}$ and $\mathcal{H} = \{\phi, \{b\}, \{c\}\}$. Consider $A = \{a\}$. Then $\land^*_\mu(A) = \{a, b\}$ and $\lor^*_\mu(A) = \{a\}$. Thus, it follows that $A$ is a $g.\land^*_\mu$-set but not a $g.\land^*_\mu$-set.
(b) Consider $X = \{a, b, c, d\}$, $\mu = \{\phi, \{a, b\}, \{b, c\}, \{a, b, c\}\}$ and $\mathcal{H} = \{\phi, \{a\}, \{c\}\}$. Let $A = \{a, c\}$. Then $\land^*_\mu(A) = \{a, b, c\}$. Thus $A$ is a $g.\land^*_\mu$-set, but not a $\land^*_\mu(A)$-set.

Theorem 8. Let $(X, \mu)$ be a GTS with a hereditary class $\mathcal{H}$ and $A \subseteq X$. Then $A$ is a $g.\lor^*_\mu$-set if and only if $U \subseteq \lor^*_\mu(A)$, whenever $U \subseteq A$ and $U$ is $\mu$-open in $X$.

Proof. Let $A$ be a $g.\lor^*_\mu$-set and $U \subseteq A$, where $U$ is $\mu$-open in $X$. Then $X\setminus A \subseteq X\setminus U$, where $X\setminus U$ is $\mu$-closed in $X$. Since $X\setminus A$ is a $g.\land^*_\mu$-set, $\land^*_\mu((X\setminus A) \subseteq X\setminus U$ which implies by Lemma 1 that $X\setminus \lor^*_\mu(A) \subseteq X\setminus U$. Thus $U \subseteq \lor^*_\mu(A)$.

Conversely, let the condition hold. Let $A$ be a subset of $X$ such that $X\setminus A \subseteq F$, where $F$ is $\mu$-closed in $X$. Then $X\setminus F \subseteq A$ and $X\setminus F$ is $\mu$-open in $X$ and so by given condition, $X\setminus F \subseteq \lor^*_\mu(A)$. Thus $X\setminus \lor^*_\mu(A) \subseteq F$ and hence by Lemma 1, $\land^*_\mu((X\setminus A) \subseteq F$. Then $X\setminus A$ is a $g.\land^*_\mu$-set. Hence $A$ is a $g.\lor^*_\mu$-set.

Theorem 9. Let $(X, \mu)$ be a GTS with a hereditary class $\mathcal{H}$ and $A \subseteq X$. If $A$ is a $g.\lor^*_\mu$-set, then $F = X$ whenever $\lor^*_\mu(A) \cup (X\setminus A) \subseteq F$ and $F$ is $\mu$-closed in $X$.

Proof. Let $A$ be a $g.\lor^*_\mu$-set and $\lor^*_\mu(A) \cup (X\setminus A) \subseteq F$, where $F$ is $\mu$-closed in $X$. Then we have $X\setminus F \subseteq X\setminus (\lor^*_\mu(A) \cup (X\setminus A)) = (X\setminus \lor^*_\mu(A)) \cap A$. Thus $X\setminus F \subseteq (X\setminus \lor^*_\mu(A))$ and $X\setminus F \subseteq A$. It follows that $X\setminus F \subseteq \lor^*_\mu(A)$ (by Theorem 8 ). Thus $X\setminus F \subseteq (X\setminus \lor^*_\mu(A)) \cup \lor^*_\mu(A) = \phi$ and hence $F = X$.

Theorem 10. Let $(X, \mu)$ be a GTS with a hereditary class $\mathcal{H}$. Then a $g.\lor^*_\mu$-set is a $\lor^*_\mu$-set if and only if $\lor^*_\mu(A) \cup (X\setminus A)$ is a $\mu$-closed set.

Proof. Suppose that a $g.\lor^*_\mu$-set $A$ is a $\lor^*_\mu$-set. Then $\lor^*_\mu(A) = A$. Thus $\lor^*_\mu(A) \cup (X\setminus A) = X \cup (X\setminus A) = X$ which is $\mu$-closed.

Conversely, let $A$ be a $g.\lor^*_\mu$-set such that $\lor^*_\mu(A) \cup (X\setminus A)$ is $\mu$-closed in $X$. Then by Theorem 9, we have $\lor^*_\mu(A) \cup (X\setminus A) = X$ and hence $A \subseteq \lor^*_\mu(A)$. Again by Theorem 4(i), $A \supseteq \lor^*_\mu(A)$. Thus $\lor^*_\mu(A) = A$ and hence $A$ is a $\lor^*_\mu$-set.

Corollary 4. Let $(X, \mu)$ be a GTS with a hereditary class $\mathcal{H}$. Then a $g.\land^*_\mu$-set is a $\land^*_\mu$-set if and only if $\land^*_\mu(A) \cup (X\setminus A)$ is $\mu$-open.

Theorem 11. Let $(X, \mu)$ be a GTS with a hereditary class $\mathcal{H}$. Then for each $x \in X$, either $\{x\}$ is a $\mu$-open set in $X$ or a $g.\lor^*_\mu$-set.

Proof. Let $x \in X$. Suppose $\{x\}$ is not $\mu$-open in $X$, then $X$ is the only $\mu$-closed set containing $X\setminus \{x\}$ and hence $X\setminus \{x\}$ is a $g.\land^*_\mu$-set. Thus $\{x\}$ is a $g.\lor^*_\mu$-set.
Proposition 1. Let \((X, \mu)\) be a GTS with a hereditary class \(\mathcal{H}\). Then every singleton of \(X\) is a \(g, \wedge^*_\mu\)-set if and only if \(U = \vee^*_\mu(U)\) for every \(\mu\)-open set \(U\) in \(X\).

Proof. Let every singleton set of \(X\) be a \(g, \wedge^*_\mu\)-set. Let \(U\) be a \(\mu\)-open set in \(X\) and \(x \in X \setminus \mu\). Since \(\{x\}\) is a \(g, \wedge^*_\mu\)-set, we have \(\wedge^*_\mu\{x\} \subseteq X \setminus \mu\). It follows that 
\[
\bigcup \{\wedge^*_\mu\{x\} : x \in X \setminus \mu\} \subseteq X \setminus \mu
\]
and thus using Theorem 3(v), we get \(\wedge^*_\mu \{\bigcup \{\{x\} : x \in X \setminus \mu\}\} \subseteq X \setminus \mu\). Therefore \(\wedge^*_\mu(X \setminus \mu) \subseteq X \setminus \mu\) and hence by Lemma 1, we have \(X \setminus \mu = \wedge^*_\mu(X \setminus \mu) = X \setminus \vee^*_\mu(U)\). Thus \(U = \vee^*_\mu(U)\).

Conversely, let \(x \in X\) and \(\{x\} \subseteq F\), where \(F\) is a \(\mu\)-closed subset of \(X\). Then \(X \setminus F\) is \(\mu\)-open in \(X\) and so by hypothesis, \(X \setminus F = \vee^*_\mu(X \setminus F) = X \setminus \wedge^*_\mu(F)\) (by Lemma 1). It follows that \(F = \wedge^*_\mu(F)\). Thus \(\wedge^*_\mu\{x\} \subseteq \wedge^*_\mu(F) = F\) and hence \(\{x\}\) is a \(g, \wedge^*_\mu\)-set.

\[\square\]

4 Applications

In [9], we introduced and studied the concept of \(\mu^*-R_0\), \(\mu^*-R_1\) and \(\mu^*-T_\frac{1}{2}\) spaces. Here we deduce some characterizations of the above lower separation axioms in terms of \(\wedge^*_\mu\) and \(\vee^*_\mu\) sets.

Definition 8. [9] A GTS \((X, \mu)\) with a hereditary class \(\mathcal{H}\) is said to be a \(\mu^*-R_0\)-space if for every \(\mu^*-\)-open set \(U\) and each \(x \in U\), one has \(c_\mu(\{x\}) \subseteq U\) that is, every singleton is \(\mu^*-\)-closed.

Definition 9. [9] A GTS \((X, \mu)\) with a hereditary class \(\mathcal{H}\) is said to be \(\mu^*-R_1\) if for each \(x, y \in X\) with \(c_\mu(\{x\}) \neq c^*_\mu(\{y\})\), there exist two disjoint \(\mu^*-\)-open sets \(U\) and \(V\) such that \(c_\mu(\{x\}) \subseteq U\) and \(c^*_\mu(\{y\}) \subseteq V\).

Proposition 1. [9] Let \((X, \mu)\) be a GTS with a hereditary class \(\mathcal{H}\). If it is \(\mu^*-R_1\), then it is also \(\mu^*-R_0\).

Theorem 13. (Theorem 3.2 of [9]) Let \((X, \mu)\) be a GTS with a hereditary class \(\mathcal{H}\). Then the following are equivalent:

(a) A GTS \((X, \mu)\) with a hereditary class \(\mathcal{H}\) is \(\mu^*-R_0\)-space.
(b) \(x \in c^*_\mu(\{y\})\) if and only if \(y \in c_\mu(\{x\})\), where \(x\) and \(y\) are any two distinct points of \(X\).

Theorem 14. (Theorem 3.3 of [9]) For a GTS \((X, \mu)\) with a hereditary class \(\mathcal{H}\), the following are equivalent:

(a) A GTS \((X, \mu)\) with a hereditary class \(\mathcal{H}\) is \(\mu^*-R_0\)-space.
(b) If \(x\) and \(y\) are two distinct points of \(X\) then \(x \notin c^*_\mu(\{y\}) \Rightarrow c_\mu(\{x\}) \cap c^*_\mu(\{y\}) = \phi\).
(c) Every \(\mu^*-\)-closed set \(F\) can be written as \(F = \cap \{U : U \text{ is }\mu\text{-open and } F \subseteq U\}\).

Theorem 15. In a \(\mu^*-R_0\)-space, for any two points \(x, y\) in \(X\), the following are equivalent:

(a) \(c_\mu(\{x\}) \neq c_\mu(\{y\})\)
(b) \(c^*_\mu(\{x\}) \neq c^*_\mu(\{y\})\)
(c) either \(c_\mu(\{x\}) \cap c^*_\mu(\{y\}) = \phi\) or \(c^*_\mu(\{x\}) \cap c_\mu(\{y\}) = \phi\).
Proof. (a) ⇒ (b) : Let \( x, y \in X \) be such that \( c_\mu(\{x\}) \neq c_\mu(\{y\}) \). Then either \( x \notin c_\mu(\{y\}) \) or \( y \notin c_\mu(\{x\}) \). If \( x \notin c_\mu(\{y\}) \) then \( x \notin c_\mu^*(\{y\}) \) and hence \( c_\mu(\{x\}) \neq c_\mu^*(\{y\}) \). If \( y \notin c_\mu(\{x\}) \) then \( c_\mu(\{x\}) \neq c_\mu^*(\{y\}) \).

(b) ⇒ (c) : Let \( x, y \in X \) be such that \( c_\mu(\{x\}) \neq c_\mu^*(\{y\}) \). Then either \( x \notin c_\mu(\{y\}) \) or \( y \notin c_\mu(\{x\}) \). If \( x \notin c_\mu^*(\{y\}) \) then by Theorem 14(a) \( x \in \wedge_\mu^* \{ \{ x \} \} \). Now \( c_\mu(\{x\}) \cap c_\mu^*(\{y\}) = \emptyset \). Next, suppose \( y \notin c_\mu(\{x\}) \) which implies that \( y \notin c_\mu^*(\{x\}) \) and hence again by Theorem 14((a) ⇔ (b)), \( c_\mu(\{y\}) \cap c_\mu^*(\{x\}) = \emptyset \).

(c) ⇒ (a) : Let \( x, y \in X \) be such that either \( c_\mu(\{x\}) \cap c_\mu^*(\{y\}) = \emptyset \) or \( c_\mu^*(\{x\}) \cap c_\mu(\{y\}) = \emptyset \). If \( c_\mu(\{x\}) \cap c_\mu^*(\{y\}) = \emptyset \) then \( y \notin c_\mu(\{x\}) \) and hence \( c_\mu(\{x\}) \neq c_\mu^*(\{y\}) \). Next if \( c_\mu^*(\{x\}) \cap c_\mu(\{y\}) = \emptyset \) then \( x \notin c_\mu(\{y\}) \) and thus \( c_\mu(\{x\}) \neq c_\mu^*(\{y\}) \).

Theorem 16. For a GTS \((X, \mu)\) with a hereditary class \(\mathcal{H}\) the following are equivalent:

(a) A GTS \((X, \mu)\) with a hereditary class \(\mathcal{H}\) is a \(\mu^*-R_0\) space.

(b) If \( F \) is \(\mu^*-\text{closed} \) and \( x \in F \), then \( \wedge^*_\mu(\{x\}) \subseteq F \) and \( F = \wedge^*_\mu(F) = \wedge^*_\mu(F) \).

(c) If \( x \in X \), then \( \wedge^*_\mu(\{x\}) = c_\mu(\{x\}) \).

Proof. (a) ⇒ (b) : Let \( F \) be \(\mu^*-\text{closed} \) and \( x \in F \). We first prove that \( F = \wedge^*_\mu(F) = \wedge^*_\mu(F) \). By (a) of Theorem 14 and Definition 1, we have \( F = \wedge^*_\mu(F) \). Again obviously \( \wedge^*_\mu(F) \subseteq \wedge^*_\mu(F) \) and by Theorem 3, \( F \subseteq \wedge^*_\mu(F) \). Thus \( F = \wedge^*_\mu(F) \) and hence \( F = \wedge^*_\mu(F) = \wedge^*_\mu(F) \). \( \square \)

(b) ⇒ (c) : Let \( x \in X \). Then \( x \in c_\mu^*(\{x\}) \) where \( c_\mu^*(\{x\}) \) is a \(\mu^*-\text{closed} \) set and so by (b), \( \wedge^*_\mu(\{x\}) \subseteq c_\mu^*(\{x\}) \). Thus \( \wedge^*_\mu(\{x\}) \subseteq c_\mu(\{x\}) \). Next we show that \( c_\mu(\{x\}) \subseteq \wedge^*_\mu(\{x\}) \). For that, let \( y \notin \wedge^*_\mu(\{x\}) \). Then there exists \( V \in \mu^* \) such that \( x \in V \) and \( y \notin V \). So \( c_\mu^*(\{y\}) \cap V = \emptyset \). By (b), we have \( c_\mu^*(\{y\}) = \wedge^*_\mu(c_\mu^*(\{y\})) = \cap \{ G \in \mu : c_\mu^*(\{y\}) \subseteq G \} \). Thus \( \cap \{ G \in \mu : c_\mu^*(\{y\}) \subseteq G \} \cap V = \emptyset \).

(c) ⇒ (a) : Let \( x, y \in X \) with \( x \neq y \). Then \( x \in c_\mu^*(\{y\}) \) if and only if \( y \in \wedge^*_\mu(\{x\}) \) and hence \( c_\mu^*(\{x\}) \subseteq c_\mu(\{x\}) \). Thus \( c_\mu^*(\{x\}) \subseteq c_\mu^*(\{x\}) \). By using Theorem 2 i.e., \( x \in c_\mu^*(\{x\}) \) if and only if \( y \in c_\mu(\{x\}) \). Hence by Theorem 13, \((X, \mu)\) with a hereditary class \(\mathcal{H}\) is a \(\mu^*-\text{R}_0\) space. \( \square \)

Theorem 17. The following are equivalent for a GTS \((X, \mu)\) with a hereditary class \(\mathcal{H}\):

(a) \((X, \mu)\) with a hereditary class \(\mathcal{H}\) is a \(\mu^*-\text{R}_0\) space.

(b) \( x \in \wedge^*_\mu(\{y\}) \) if and only if \( y \in \wedge^*_\mu(\{x\}) \), for any two distinct points \( x, y \in X \).

Proof. (a) ⇒ (b) : Suppose that \((X, \mu)\) with a hereditary class \(\mathcal{H}\) is a \(\mu^*-\text{R}_0\) space. Let \( x, y \in X \) with \( x \neq y \). First let \( x \in \wedge^*_\mu(\{y\}) \). Then by Theorem 2, \( y \in c_\mu^*(\{x\}) \). We now show that \( y \in \wedge^*_\mu(\{x\}) \). Indeed, if \( y \notin \wedge^*_\mu(\{x\}) \) then there exists \( V \in \mu \) such that \( x \in V \) and \( y \notin V \) and so \( c_\mu(\{y\}) \cap V = \emptyset \). Thus \( \cap \{ G \in \mu : c_\mu(\{y\}) \subseteq G \} \cap V = \emptyset \) (refer to Theorem 14(a) ⇒ (c)) and hence there exists \( G \in \mu \) such that \( x \notin G \) and \( c_\mu(\{y\}) \subseteq G \) and hence \( c_\mu(\{x\}) \cap G = \emptyset \). This implies that...
Theorem 18. A GTS \((X, \mu)\) with a hereditary class \(\mathcal{H}\) is \(\mu^*-R_1\) if and only if for any two distinct points \(x, y \in X\) with \(\land^*_\mu(\{x\}) \neq \land_\mu(\{y\})\), there exist two disjoint \(\mu^*-\)open sets \(U\) and \(V\) such that \(c_\mu(\{x\}) \subseteq U\) and \(c^*_\mu(\{y\}) \subseteq V\).

Proof. Suppose that \((X, \mu)\) with a hereditary class \(\mathcal{H}\) is \(\mu^*-R_1\). Let \(x, y\) be any two distinct points in \(X\) such that \(\land^*_\mu(\{x\}) \neq \land_\mu(\{y\})\). Then we have either \(x \notin \land_\mu(\{y\})\) or \(y \notin \land^*_\mu(\{x\})\). If not then \(x \in \land_\mu(\{y\})\) and \(y \in \land^*_\mu(\{x\})\) and so \(\land_\mu(\{x\}) \subseteq \land_\mu(\land^*_\mu(\{x\})) = \land_\mu(\{y\})\) and thus \(\land^*_\mu(\{x\}) \subseteq \land_\mu(\{y\})\). By hypothesis, we have \(\land^*_\mu(\{x\}) \not\subset \land_\mu(\{y\})\). Since \((X, \mu)\) with the hereditary class \(\mathcal{H}\) is \(\mu^*-R_1\), by Proposition 1, it is a \(\mu^*-R_0\) space. Now by using \(((a) \Rightarrow (c))\) of Theorem 16, we have \(c_\mu(\{x\}) \not\subset \land_\mu(\{y\})\)\(\ldots(1)\). Let \(z \in \land_\mu(\{y\})\). Then by Theorem 1, we have \(y \in c_\mu(\{z\})\). It follows that \(c^*_\mu(\{y\}) \subseteq c_\mu(\{z\})\) and hence by Theorem 15 \(((b) \Rightarrow (c))\), we must have \(c^*_\mu(\{y\}) = c_\mu(\{z\})\). Thus \(z \in c^*_\mu(\{y\})\) and hence \(\land_\mu(\{y\}) \subseteq c^*_\mu(\{y\})\)\(\ldots(2)\). From \((1)\) and \((2)\), we get \(c_\mu(\{x\}) \not\subset c^*_\mu(\{y\})\) and hence there are no two disjoint \(\mu^*-\)open sets \(U\) and \(V\) such that \(c_\mu(\{x\}) \subseteq U\) and \(c^*_\mu(\{y\}) \subseteq V\). This contradicts the fact that the space is \(\mu^*-R_1\).

Now we show that \(c_\mu(\{x\}) \neq c^*_\mu(\{y\})\).
Case I: If \(x \notin \land_\mu(\{y\})\) then \(y \notin c_\mu(\{x\})\) which implies \(c_\mu(\{x\}) \neq c^*_\mu(\{y\})\).
Case II: If \(y \notin \land^*_\mu(\{x\})\) then \(x \notin c^*_\mu(\{y\})\) which implies \(c_\mu(\{x\}) \neq c^*_\mu(\{y\})\).
Thus, in both the cases, we have \(c_\mu(\{x\}) \neq c^*_\mu(\{y\})\). Then by definition of \(\mu^*-R_1\) space, there exist two disjoint \(\mu^*-\)open sets \(U\) and \(V\) such that \(c_\mu(\{x\}) \subseteq U\) and \(c^*_\mu(\{y\}) \subseteq V\).

Conversely, let the condition hold. Let \(x\) and \(y\) be two distinct points of \(X\) such that \(c_\mu(\{x\}) \neq c^*_\mu(\{y\})\). Then either \(y \notin c_\mu(\{x\})\) or \(x \notin c^*_\mu(\{y\})\).
Case I: If \(y \notin c_\mu(\{x\})\) then there exists a \(\mu\)-open set \(G\) containing \(y\) such that \(x \notin G\) and hence from the definition \(x \notin \land_\mu(\{y\})\) which follows that \(\land_\mu(\{y\}) \neq \land^*_\mu(\{x\})\).
Case II: If \(x \notin c^*_\mu(\{y\})\) then by Theorem 2, \(y \notin \land^*_\mu(\{x\})\) which shows that \(\land^*_\mu(\{x\}) \neq \land_\mu(\{y\})\).
Thus, in both the cases, we have \(\land^*_\mu(\{x\}) \neq \land_\mu(\{y\})\) and hence by hypothesis, there exist two disjoint \(\mu^*-\)open sets \(U\) and \(V\) such that \(c_\mu(\{x\}) \subseteq U\) and \(c^*_\mu(\{y\}) \subseteq V\). This shows that \((X, \mu)\) with a hereditary class \(\mathcal{H}\) is \(\mu^*-R_1\).

Definition 10. An \((X, \mu)\) with a hereditary class \(\mathcal{H}\) is said to be \(\mu^*-T_1\) if every \(\mu^*-g\)-closed set is \(\mu\)-closed in \(X\).

Theorem 19. An \((X, \mu)\) with a hereditary class \(\mathcal{H}\) is a \(\mu^*-T_\frac{1}{2}\) space if and only if for each \(x \in X\), either \(\{x\}\) is \(\mu^*-\)closed or \(\mu\)-open in \(X\).
Theorem 20. Let \((X, \mu)\) be a GTS with a hereditary class \(\mathcal{H}\). Then it is a \(\mu^*-T_{\frac{1}{2}}\)-space if and only if every \(g.\mathcal{V}_{\mu}^*\)-set is a \(\mathcal{V}_{\mu}^*\)-set.

Proof. Let a GTS \((X, \mu)\) with a hereditary class \(\mathcal{H}\) be a \(\mu^*-T_{\frac{1}{2}}\)-space. We prove by contradiction. Suppose that \(A\) is a \(g.\mathcal{V}_{\mu}^*\)-set but not a \(\mathcal{V}_{\mu}^*\)-set. Then there exists an element \(x \in A\) such that \(x \notin \mathcal{V}_{\mu}^*(A)\). Thus by definition of \(\mathcal{V}_{\mu}^*(A)\), \(\{x\}\) is not \(\mu^*\)-closed. Thus by Theorem 19, we have \(\{x\}\) is \(\mu\)-open, that is, \(X\setminus\{x\}\) is \(\mu\)-closed in \(X\). Since \(x \in A\) and \(x \notin \mathcal{V}_{\mu}^*(A)\), we have \(\mathcal{V}_{\mu}^*(A) \cup (X\setminus A) \subseteq X\setminus\{x\}\).

Therefore by Theorem 9, \(X\setminus\{x\} = X\), a contradiction. Conversely, let every \(g.\mathcal{V}_{\mu}^*\)-set be a \(\mathcal{V}_{\mu}^*\)-set. Suppose the GTS \((X, \mu)\) with a hereditary class \(\mathcal{H}\) is not a \(\mu^*-T_{\frac{1}{2}}\)-space. Then there exists a \(\mu^*\)-g-closed set \(A\) which is not \(\mu\)-closed in \(X\). Thus, there exists an element \(x \in X\) such that \(x \in c_{\mu}(A)\) but \(x \notin A\). Now by Theorem 11, \(\{x\}\) is either a \(\mu\)-open set or a \(g.\mathcal{V}_{\mu}^*\)-set.

Case I: Let \(\{x\}\) be \(\mu\)-open. Then \(x \in c_{\mu}(A)\) implies that \(\{x\} \cap A \neq \phi\) and so \(x \in A\), a contradiction.

Case II: Let \(\{x\}\) be a \(g.\mathcal{V}_{\mu}^*\)-set. Then by hypothesis \(\{x\}\) is a \(\mathcal{V}_{\mu}^*\)-set and so \(\{x\} = \mathcal{V}_{\mu}^*(\{x\})\) and hence by definition of \(\mathcal{V}_{\mu}^*\)-set, we have \(\{x\}\) is \(\mu^*\)-closed. Since \(A\) is \(\mu^*\)-g-closed and \(A \subseteq X\setminus\{x\}\), \(c_{\mu}(A) \subseteq X\setminus\{x\}\), which contradict that \(x \in c_{\mu}(A)\).

Hence \((X, \mu)\) with a hereditary class \(\mathcal{H}\) is a \(\mu^*-T_{\frac{1}{2}}\)-space. \(\square\)

Corollary 5. Let \((X, \mu)\) be a GTS with a hereditary class \(\mathcal{H}\). Then it is a \(\mu^*-T_{\frac{1}{2}}\)-space if and only if every \(g.\mathcal{L}_{\mu}^*\)-set is a \(\mathcal{L}_{\mu}^*\)-set.

Corollary 6. Let \((X, \mu)\) be a GTS with a hereditary class \(\mathcal{H}\). Then it is a \(\mu^*-T_{\frac{1}{2}}\)-space if and only if for each \(x \in X\), either \(\{x\}\) is a \(\mathcal{V}_{\mu}^*\)-set or \(\mu\)-open in \(X\).

Proof. Let \(X\) be a \(\mu^*-T_{\frac{1}{2}}\)-space. Now we have from Theorem 11 that for each \(x \in X\), either \(\{x\}\) is a \(\mu\)-open set or a \(g.\mathcal{V}_{\mu}^*\)-set in \(X\). If \(\{x\}\) is \(\mu\)-open in \((X, \mu)\) then we are done. So suppose that \(\{x\}\) is not a \(\mu\)-open set in \(X\). Then it must be a \(g.\mathcal{V}_{\mu}^*\)-set and so by Theorem 20, we get \(\{x\}\) is a \(\mathcal{V}_{\mu}^*\)-set.

Conversely, let \(x \in X\). Then either \(\{x\}\) is a \(\mathcal{V}_{\mu}^*\)-set or \(\mu\)-open in \(X\).

Case I: If \(\{x\}\) is a \(\mathcal{V}_{\mu}^*\)-set in \(X\), then \(\{x\} = \mathcal{V}_{\mu}^*(\{x\})\) and so by definition of \(\mathcal{V}_{\mu}^*\)-set, we have \(\{x\}\) is \(\mu^*\)-closed. Hence by Theorem 19, \(X\) is a \(\mu^*-T_{\frac{1}{2}}\)-space.

Case II: If \(\{x\}\) is \(\mu\)-open in \(X\), then by Theorem 19, \(X\) is a \(\mu^*-T_{\frac{1}{2}}\)-space. \(\square\)

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References


New types of sets in GTS


