UNIVALENCE CRITERIA FOR A GENERAL INTEGRAL OPERATOR

Constantin Lucian ALDEA and Virgil PESCAR

Abstract

In this work we consider a general integral operator and we derive conditions for the univalence of this integral operator.

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1 Introduction

Let $A$ be the class of the functions $f$ which are analytic in the open unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$ and $f(0) = f'(0) - 1 = 0$.

We denote by $S$ the subclass of $A$ consisting of functions $f \in A$, which are univalent in $U$.

Let $P$ denote the class of functions $p$ which are analytic in $U$, $p(0) = 1$ and $Re\, p(z) > 0$, for all $z \in U$.

In [7] Pescar introduced a general integral operator

$$I_n(z) = \left[ \delta \int_0^z u^{\delta - 1} \left( \frac{f_1(u)}{g_1(u)} \right)^{\alpha_1} \cdots \left( \frac{f_n(u)}{g_n(u)} \right)^{\alpha_n} \left( \frac{f'_1(u)}{g'_1(u)} \right)^{\beta_1} \cdots \left( \frac{f'_n(u)}{g'_n(u)} \right)^{\beta_n} \right]^{\frac{1}{\delta}}, \quad (1)$$

for $f_j, g_j \in A$ and complex numbers $\delta, \alpha_j, \beta_j$ ($\delta \neq 0$), $j = 1, n$, $n \in \mathbb{N} - \{0\}$.

We have the next particular cases:

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1Faculty of Mathematics and Informatics, Transilvania University of Brașov, Romania, e-mail: costel.aldea@unitbv.ro
2Faculty of Mathematics and Informatics, Transilvania University of Brașov, Romania, e-mail: virgilpescar@unitbv.ro
1. From (1), for \( \delta = \beta, \alpha_1 = \gamma, \alpha_i = 0, i = \overline{2,n}, \beta_i = 0, i = \overline{1,n}, \) and \( g_1(z) = z, \) \( h(z) \equiv f_1(z), \) \( z \in \mathbb{U}, \) we obtain the integral operator

\[
J_{\beta,\gamma}(z) = \left[ \beta \int_{0}^{z} u^{\beta-1} \left( \frac{h(u)}{u} \right)^{\gamma} du \right]^{\frac{1}{\beta}},
\]

which was defined by Pascu and Pescar [6], in the year 1990.

2. For \( \beta_1 = \beta_2 = \cdots = \beta_n = 0, \) from (1) we have the integral operator

\[
G_n(z) = \left[ \delta \int_{0}^{z} u^{\delta-1} \left( \frac{f_1(u)}{g_1(u)} \right)^{\alpha_1} \cdots \left( \frac{f_n(u)}{g_n(u)} \right)^{\alpha_n} du \right]^{\frac{1}{\delta}},
\]

for \( \delta \) a complex number, \( \delta \neq 0 \) and \( f_j, g_j \in A, j = \overline{1,n}, \) introduced by Moldoveanu, Ovesea and Pascu [4] in the year 1991.

3. For \( \beta_i = 0, i = \overline{1,n} \) and \( g_i(z) = z, i = \overline{1,n}, \) from (1) we obtain the integral operator

\[
D_n(z) = \left[ \delta \int_{0}^{z} u^{\delta-1} \left( \frac{f_1(u)}{u} \right)^{\alpha_1} \cdots \left( \frac{f_n(u)}{u} \right)^{\alpha_n} du \right]^{\frac{1}{\delta}},
\]

\( \alpha_i, \delta \) complex numbers, \( i = \overline{1,n}, \delta \neq 0, \) defined by Breaz, D. and Breaz, N. [1], in the year 2002.

This integral operator is the particular case of the integral operator \( G_n, \) for \( g_j(z) = z, j = \overline{1,n}. \)

4. From (1), for \( g_i(z) = z, i = \overline{1,n} \) we obtain the general integral operator

\[
F_n(z) = \left[ \delta \int_{0}^{z} f_1'(u) \left( \frac{f_1(u)}{u} \right)^{\alpha_1} \cdots f_n'(u) \left( \frac{f_n(u)}{u} \right)^{\alpha_n} du \right]^{\frac{1}{\delta}},
\]

defined by Frasin [2], in the year 2011.

Properties of certain integral operators were studied by different authors in the following papers [8, 9, 10, 11, 12, 13].

In this paper we obtain the univalence criteria for the integral operator \( I_n. \)

2 Preliminary results

We need the following lemmas.

**Lemma 1.** ([5]). Let \( \gamma, \delta \) be complex numbers, \( \text{Re} \gamma > 0 \) and \( f \in A. \) If

\[
\frac{1 - |z|^{2 \text{Re} \gamma}}{\text{Re} \gamma} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1,
\]

(1)
for all $z \in U$, then for any complex number $\delta$, $\text{Re} \, \delta \geq \text{Re} \, \gamma$, the function $F_\delta$ defined by

$$F_\delta(z) = \left[ \delta \int_0^z u^{\delta-1} f'(u)du \right]^{1/\delta} \quad (2)$$

is regular and univalent in $U$.

**Lemma 2. (Schwarz [3])** Let $f$ be the function regular in the disk $U_R = \{ z \in \mathbb{C} : |z| < R \}$ with $|f(z)| < M$, $M$ fixed. If $f(z)$ has in $z = 0$ one zero with multiply $\geq m$, then

$$|f(z)| \leq \frac{M}{R^m} |z|^m, \quad (z \in U_R), \quad (3)$$

the equality (in the inequality (3) for $z \neq 0$) can hold only if

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

where $\theta$ is constant.

### 3 Main results

**Theorem 1.** Let $\gamma, \delta, \alpha_j, \beta_j$ be complex numbers, $c = \text{Re} \, \gamma > 0$, $j = \overline{1,n}$, $M_{ij}, L_{ij}$ real positive numbers, $i = \overline{1,2}$, $j = \overline{1,n}$ and $f_j, g_j \in A$, $f_j(z) = z + a_{2j}z^2 + a_{3j}z^3 + \ldots$, $g_j(z) = z + b_{2j}z^2 + b_{3j}z^3 + \ldots$, $j = \overline{1,n}$.

If

$$\left| \frac{zf_j'(z)}{f_j(z)} - 1 \right| \leq M_{1j}, \quad (1)$$

$$\left| \frac{zg_j'(z)}{g_j(z)} - 1 \right| \leq M_{2j}, \quad (2)$$

$$\left| \frac{zf_j''(z)}{f_j'(z)} \right| \leq L_{1j}, \quad (3)$$

$$\left| \frac{zg_j''(z)}{g_j'(z)} \right| \leq L_{2j}, \quad (4)$$

for all $z \in U$, $j = \overline{1,n}$ and

$$\sum_{j=1}^n [(M_{1j} + M_{2j})|\alpha_j| + (L_{1j} + L_{2j})|\beta_j|] \leq \frac{(2c + 1)^{2c+1}}{2c}, \quad (5)$$

then for all $\delta$ complex numbers, $\text{Re} \, \delta \geq \text{Re} \, \gamma$, the integral operator $I_n$, given by (1) is in the class $S$. 

Proof. Let us consider the function

\[ h_n(z) = \int_0^z \left( \frac{f_1(u)}{g_1(u)} \right)^{\alpha_1} \cdots \left( \frac{f_n(u)}{g_n(u)} \right)^{\alpha_n} \left( \frac{f'_1(u)}{g'_1(u)} \right)^{\beta_1} \cdots \left( \frac{f'_n(u)}{g'_n(u)} \right)^{\beta_n} \, du, \]

(6)

for \( f_j, g_j \in A, j = 1, n. \)

The function \( h_n \) is regular in \( U \) and \( h(0) = h'(0) - 1 = 0. \)

We have:

\[ zh'_n(z) = \sum_{j=1}^n \left[ \alpha_j \left( \frac{zf'_j(z)}{f_j(z)} - \frac{zg'_j(z)}{g_j(z)} \right) + \beta_j \left( \frac{zf''_j(z)}{f'_j(z)} - \frac{zg''_j(z)}{g'_j(z)} \right) \right], \]

(7)

for all \( z \in U \), and hence, we get

\[ \frac{zh''_n(z)}{h'_n(z)} = \sum_{j=1}^n \left\{ \alpha_j \left[ \left( \frac{zf'_j(z)}{f_j(z)} - 1 \right) - \left( \frac{zg'_j(z)}{g_j(z)} - 1 \right) \right] + \right. \]

\[ \left. + \beta_j \left( \frac{zf''_j(z)}{f'_j(z)} - \frac{zg''_j(z)}{g'_j(z)} \right) \right\} \]

(8)

for all \( z \in U \).

From (8) we obtain

\[ 1 - |z|^{2c} \left| \frac{zh''_n(z)}{h'_n(z)} \right| \leq \frac{1 - |z|^{2c}}{c} \left\{ \sum_{j=1}^n \left| \alpha_j \left( \frac{zf'_j(z)}{f_j(z)} - 1 \right) + \frac{zg'_j(z)}{g_j(z)} - 1 \right) \right] + \]

\[ + |\beta_j| \left( \frac{|zf''_j(z)|}{|f'_j(z)|} + \frac{|zg''_j(z)|}{|g'_j(z)|} \right) \left\} \right. \]

(9)

for all \( z \in U \).

Applying Lemma 2 we get

\[ \left| \frac{zf'_j(z)}{f_j(z)} - 1 \right| \leq M_{1j}|z|, \]

(10)

\[ \left| \frac{zg'_j(z)}{g_j(z)} - 1 \right| \leq M_{2j}|z|, \]

(11)

\[ \left| \frac{zf''_j(z)}{f'_j(z)} \right| \leq L_{1j}|z|, \]

(12)

\[ \left| \frac{zg''_j(z)}{g'_j(z)} \right| \leq L_{2j}|z|, \]

(13)
for all $z \in \mathbb{U}$, $j = 1, n$.

Using these inequalities from (9) we have

$$
\frac{1 - |z|^{2c}}{c} \left| \frac{z h''_n(z)}{h'_n(z)} \right| \leq \frac{1 - |z|^{2c}}{c} |z| \left\{ \sum_{j=1}^{n} [(M_{1j} + M_{2j})|\alpha_j| + (L_{1j} + L_{2j})|\beta_j|] \right\}
$$

(14)

for all $z \in \mathbb{U}$.

Since

$$
\max_{|z| \leq 1} \left( 1 - |z|^{2c} \right) = \frac{2}{(2c + 1)\frac{2c+1}{2c}},
$$

from (14) we obtain

$$
\frac{1 - |z|^{2c}}{c} \left| \frac{z h''_n(z)}{h'_n(z)} \right| \leq \frac{2}{(2c + 1)\frac{2c+1}{2c}} \left\{ \sum_{j=1}^{n} [(M_{1j} + M_{2j})|\alpha_j| + (L_{1j} + L_{2j})|\beta_j|] \right\}
$$

and hence, by (5) we have

$$
\frac{1 - |z|^{2c}}{c} \left| \frac{z h''_n(z)}{h'_n(z)} \right| \leq 1, \quad (z \in \mathbb{U}).
$$

(15)

From (6) we obtain

$$
h'_n(z) = \left( \frac{f_1(z)}{g_1(z)} \right)^{\alpha_1} \cdots \left( \frac{f_n(z)}{g_n(z)} \right)^{\alpha_n} \left( \frac{f'_1(z)}{g'_1(z)} \right)^{\beta_1} \cdots \left( \frac{f'_n(z)}{g'_n(z)} \right)^{\beta_n}
$$

and using (15), by Lemma 1, it results that the integral operator $I_n$ given by (1) is in the class $S$.

\[\Box\]

**Corollary 1.** Let $\gamma, \alpha_j, \beta_j$ be complex numbers, $0 < Re \gamma \leq 1$, $c = Re \gamma$, $M_{ij}, L_{ij}$ positive real numbers, $i = 1, 2$, $j = 1, n$ and $f_j, g_j \in \mathcal{A}$, $f_j(z) = z + a_{2j}z^2 + a_{3j}z^3 + \ldots$, $g_j(z) = z + b_{2j}z^2 + b_{3j}z^3 + \ldots$, $j = 1, n$.

If

$$
\left| \frac{zf'_j(z)}{f_j(z)} - 1 \right| \leq M_{1j},
$$

(16)

and

$$
\left| \frac{zg'_j(z)}{g_j(z)} - 1 \right| \leq M_{2j},
$$

(17)
\[ \left| \frac{zf_j''(z)}{f_j'(z)} \right| \leq L_{1j}, \quad (18) \]

\[ \left| \frac{zg_j''(z)}{g_j'(z)} \right| \leq L_{2j}, \quad (19) \]

for all \( z \in \mathcal{U}, \ j = 1, n \) and

\[ \sum_{j=1}^{n} [(M_{1j} + M_{2j})|\alpha_j| + (L_{1j} + L_{2j})|\beta_j|] \leq \frac{(2c + 1)^{2c+1}}{2}, \quad (20) \]

then the integral operator \( T_n \) defined by

\[ T_n(z) = \int_{0}^{z} \left( \frac{f_1(u)}{g_1(u)} \right)^{\alpha_1} \cdots \left( \frac{f_n(u)}{g_n(u)} \right)^{\alpha_n} \left( \frac{f_1'(u)}{g_1'(u)} \right)^{\beta_1} \cdots \left( \frac{f_n'(u)}{g_n'(u)} \right)^{\beta_n} du, \quad (21) \]

is in the class \( S \).

**Proof.** For \( \delta = 1 \), from Theorem 1, we obtain Corollary 1.

**Corollary 2.** Let \( \gamma, \alpha_j \) be complex numbers, \( 0 < \Re \gamma \leq 1, \ c = \Re \gamma, \ M_{ij} \) positive real numbers, \( i = 1, 2, \ j = 1, n \) and \( f_j, g_j \in \mathcal{A}, \)

\( f_j(z) = z + a_{2j}z^2 + a_{3j}z^3 + \ldots, \ g_j(z) = z + b_{2j}z^2 + b_{3j}z^3 + \ldots, \ j = 1, n. \)

If

\[ \left| \frac{zf_j'(z)}{f_j(z)} - 1 \right| \leq M_{1j}, \quad (22) \]

\[ \left| \frac{zg_j'(z)}{g_j(z)} - 1 \right| \leq M_{2j}, \quad (23) \]

for all \( z \in \mathcal{U}, \ j = 1, n \) and

\[ \sum_{j=1}^{n} (M_{1j} + M_{2j})|\alpha_j| \leq \frac{(2c + 1)^{2c+1}}{2}, \quad (24) \]

then the integral operator \( H_n \) defined by

\[ H_n(z) = \int_{0}^{z} \left( \frac{f_1(u)}{g_1(u)} \right)^{\alpha_1} \cdots \left( \frac{f_n(u)}{g_n(u)} \right)^{\alpha_n} du, \quad (25) \]

is in the class \( S \).

**Proof.** For \( \delta = 1 \) and \( \beta_1 = \beta_2 = \cdots = \beta_n = 0 \), from Theorem 1, we have the Corollary 2. \( \square \)
Corollary 3. Let $\gamma, \beta_j$ be complex numbers, $0 < \Re \gamma \leq 1$, $c = \Re \gamma$, $L_{ij}$ positive real numbers, $i = 1, 2$, $j = 1, n$ and $f_j, g_j \in A$, $f_j(z) = z + a_{2j}z^2 + a_{3j}z^3 + \ldots$, $g_j(z) = z + b_{2j}z^2 + b_{3j}z^3 + \ldots$, $j = 1, n$.

If
\[ \left\lvert \frac{zf''_j(z)}{f'_j(z)} \right\rvert \leq L_{1j}, \quad (26) \]
\[ \left\lvert \frac{zg''_j(z)}{g'_j(z)} \right\rvert \leq L_{2j}, \quad (27) \]
for all $z \in \mathbb{U}$, $j = 1, n$ and
\[ \sum_{j=1}^{n} (L_{1j} + L_{2j})|\beta_j| \leq \frac{(2c+1)2^c+1}{2}, \quad (28) \]
then the integral operator $K_n$ defined by
\[ K_n(z) = \int_0^z \left( \frac{f'_1(u)}{g'_1(u)} \right)^{\beta_1} \cdots \left( \frac{f'_n(u)}{g'_n(u)} \right)^{\beta_n} du, \quad (29) \]
is in the class $S$.

Proof. We take $\delta = 1$, $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$, in Theorem 1. \qed

Corollary 4. Let $\gamma, \delta, \alpha_j$, be complex numbers, $c = \Re \gamma > 0$, $\Re \delta \geq \Re \gamma$, $j = 1, n$, $M_{ij}$ real positive numbers, $i = 1, 2$, $j = 1, n$ and $f_j, g_j \in A$, $f_j(z) = z + a_{2j}z^2 + a_{3j}z^3 + \ldots$, $g_j(z) = z + b_{2j}z^2 + b_{3j}z^3 + \ldots$, $j = 1, n$.

If
\[ \left\lvert \frac{zf'_j(z)}{f_j(z)} - 1 \right\rvert \leq M_{1j}, \quad (30) \]
\[ \left\lvert \frac{zg'_j(z)}{g_j(z)} - 1 \right\rvert \leq M_{2j}, \quad (31) \]
for all $z \in \mathbb{U}$, $j = 1, n$ and
\[ \sum_{j=1}^{n} [(M_{1j} + M_{2j})|\alpha_j|] \leq \frac{(2c+1)2^{c+1}}{2}, \quad (32) \]
then the integral operator $G_n$ defined by
\[ G_n(z) = \left[ \delta \int_0^z u^{\delta-1} \left( \frac{f'_1(u)}{g'_1(u)} \right)^{\alpha_1} \cdots \left( \frac{f'_n(u)}{g'_n(u)} \right)^{\alpha_n} du \right]^\frac{1}{\delta}, \quad (33) \]
is in the class $S$. 

Univalence criteria for a general integral operator
Proof. For $\beta_1 = \beta_2 = \cdots = \beta_n = 0$, from Theorem 1, we obtain Corollary 4. □

**Corollary 5.** Let $\gamma, \delta, \beta_j$ be complex numbers, $c = \text{Re } \gamma > 0$, $\text{Re } \delta \geq \text{Re } \gamma$, $j = 1, n$, $L_{ij}$ real positive numbers, $i = 1, 2$, $j = 1, n$ and $f_j, g_j \in \mathcal{A}$, $f_j(z) = z + a_2 j z^2 + a_3 j z^3 + \ldots$, $g_j(z) = z + b_2 j z^2 + b_3 j z^3 + \ldots$, $j = 1, n$.

If

$$\left| \frac{zf_j''(z)}{f_j'(z)} \right| \leq L_{1j},$$

$$\left| \frac{zg_j''(z)}{g_j'(z)} \right| \leq L_{2j},$$

for all $z \in \mathcal{U}$, $j = 1, n$ and

$$\sum_{j=1}^{n} (L_{1j} + L_{2j}) |\beta_j| \leq \frac{(2c + 1) \gamma}{2},$$

then the integral operator $Q_n$ defined by

$$Q_n(z) = \left[ \delta \int_0^z u^{\delta-1} \left( \frac{f_1'(u)}{g_1'(u)} \right)^{\beta_1} \cdots \left( \frac{f_n'(u)}{g_n'(u)} \right)^{\beta_n} \, du \right]^\frac{1}{2},$$

is in the class $S$.

Proof. We take $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ in Theorem 1. □

**Theorem 2.** Let $\gamma, \alpha_j, \beta_j$ be complex numbers, $j = 1, n$, $c = \text{Re } \gamma > 0$ and $f_j, g_j \in \mathcal{S}$, $f_j', g_j' \in \mathcal{P}$, $f_j(z) = z + a_2 j z^2 + a_3 j z^3 + \ldots$, $g_j(z) = z + b_2 j z^2 + b_3 j z^3 + \ldots$, $j = 1, n$.

If

$$2 \sum_{j=1}^{n} |\alpha_j| + \sum_{j=1}^{n} |\beta_j| \leq \frac{c}{4},$$

for $0 < c < 1$

or

$$2 \sum_{j=1}^{n} |\alpha_j| + \sum_{j=1}^{n} |\beta_j| \leq \frac{1}{4},$$

for $c \geq 1$

then for any complex numbers $\delta$, $\text{Re } \delta \geq c$, the integral operator $I_n$, defined by

(1) is in the class $S$. 

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**Proof.** For $\beta_1 = \beta_2 = \cdots = \beta_n = 0$, from Theorem 1, we obtain Corollary 4. □
Proof. We consider the function
\[
    h_n(z) = \int_0^z \left( \frac{f_1(u)}{g_1(u)} \right)^{\alpha_1} \cdots \left( \frac{f_n(u)}{g_n(u)} \right)^{\alpha_n} \left( \frac{f'_1(u)}{g'_1(u)} \right)^{\beta_1} \cdots \left( \frac{f'_n(u)}{g'_n(u)} \right)^{\beta_n} \, du,
\]
for \( f_j, g_j \in S, f'_j, g'_j \in P, j = 1, n \). The function \( h_n \) is regular in \( U \) and \( h_n(0) = h'_n(0) - 1 = 0 \). We obtain
\[
    1 - |z|^{2c} \left| \frac{zh''_n(z)}{h'_n(z)} \right| \leq 1 - |z|^{2c} \sum_{j=1}^{n} \left[ |\alpha_j| \left( \left| \frac{zf'_j(z)}{f_j(z)} \right| + \left| \frac{zg'_j(z)}{g_j(z)} \right| \right) + |\beta_j| \left( \left| \frac{zf''_j(z)}{f'_j(z)} \right| + \left| \frac{zg''_j(z)}{g'_j(z)} \right| \right) \right],
\]
for all \( z \in U \). Since \( f_j, g_j \in S \) we have
\[
    \left| \frac{zf'_j(z)}{f_j(z)} \right| \leq \frac{1 + |z|}{1 - |z|},
\]
\[
    \left| \frac{zg'_j(z)}{g_j(z)} \right| \leq \frac{1 + |z|}{1 - |z|},
\]
for all \( z \in U, j = 1, n \).

For \( f'_j, g'_j \in P \) we have
\[
    \left| \frac{zf''_j(z)}{f'_j(z)} \right| \leq \frac{2|z|}{1 - |z|^2},
\]
\[
    \left| \frac{zg''_j(z)}{g'_j(z)} \right| \leq \frac{2|z|}{1 - |z|^2},
\]
for all \( z \in U, j = 1, n \).

Using these relations we get
\[
    \frac{1 - |z|^{2c}}{c} \left| \frac{zh''_n(z)}{h'_n(z)} \right| \leq \frac{1 - |z|^{2c}}{c} \frac{2(1 + |z|)}{1 - |z|} \sum_{j=1}^{n} |\alpha_j| + \frac{4|z|}{1 - |z|^2} \sum_{j=1}^{n} |\beta_j|,
\]
for all \( z \in U \).

For \( 0 < c < 1 \), we have \( 1 - |z|^{2c} \leq 1 - |z|^2 \), \( z \in U \) and by (46) we obtain
\[
    \frac{1 - |z|^{2c}}{c} \left| \frac{zh''_n(z)}{h'_n(z)} \right| \leq \frac{8}{c} \sum_{j=1}^{n} |\alpha_j| + \frac{4}{c} \sum_{j=1}^{n} |\beta_j|, \quad (z \in U).
\]
From (38) and (47) we have
\[
1 - \left| z \right|^{2c} \frac{|zh''_n(z)|}{h'_n(z)} \leq 1 \tag{48}
\]
for all \( z \in \mathbb{U} \) and \( 0 < c \leq 1 \).

For \( c > 1 \), we have
\[
1 - \left| z \right|^{2c} \leq 1 - \left| z \right|^2, \quad z \in \mathbb{U}
\]
and using (46) we get
\[
1 - \left| z \right|^{2c} \frac{|zh''_n(z)|}{h'_n(z)} \leq 8 \sum_{j=1}^{n} |\alpha_j| + 4 \sum_{j=1}^{n} |\beta_j|, \tag{49}
\]
for all \( z \in \mathbb{U} \), \( c \geq 1 \).

From (39) and (49) we obtain
\[
1 - \left| z \right|^{2c} \frac{|zh''_n(z)|}{h'_n(z)} \leq 1 \tag{50}
\]
for all \( z \in \mathbb{U} \), \( c \geq 1 \).

Using (40) we have
\[
h'_n(z) = \left( \frac{f_1(u)}{g_1(u)} \right)^{\alpha_1} \cdots \left( \frac{f_n(u)}{g_n(u)} \right)^{\alpha_n} \left( \frac{f'_1(u)}{g'_1(u)} \right)^{\beta_1} \cdots \left( \frac{f'_n(u)}{g'_n(u)} \right)^{\beta_n},
\]
and by (48), (50) and Lemma 1 it results that the integral operator \( I_n \), given by (1), is in the class \( S \).

\[\text{Corollary 6.} \quad \text{Let} \quad \gamma, \alpha_j, \beta_j \quad \text{be complex numbers,} \quad 0 < \Re \gamma \leq 1 \quad \text{and} \quad f_j, g_j \in \mathcal{S}, \quad f'_j, g'_j \in \mathcal{P}, \quad f_j(z) = z + a_{2j}z^2 + \ldots, \quad g_j(z) = z + b_{2j}z^2 + \ldots, \quad j = 1, n.
\]

If
\[
2 \sum_{j=1}^{n} |\alpha_j| + \sum_{j=1}^{n} |\beta_j| \leq \frac{\Re \gamma}{4}, \tag{51}
\]
then the integral operator \( T_n \) defined by (21) belongs to the class \( S \).

\[\text{Proof.} \quad \text{We take} \quad \delta = 1 \quad \text{in Theorem 2.} \hspace{1cm} \square\]

\[\text{Corollary 7.} \quad \text{Let} \quad \gamma, \alpha_j \quad \text{be complex numbers,} \quad j = 1, n, \quad 0 < \Re \gamma \leq 1, \quad f_j, g_j \in \mathcal{S}, \quad f_j(z) = z + a_{2j}z^2 + \ldots, \quad g_j(z) = z + b_{2j}z^2 + \ldots, \quad j = 1, n.
\]

If
\[
\sum_{j=1}^{n} |\alpha_j| \leq \frac{\Re \gamma}{8}, \tag{52}
\]
then the integral operator \( H_n \) given by (25) is in the class \( S \).

\[\text{Proof.} \quad \text{For} \quad \delta = 1 \quad \text{and} \quad \beta_1 = \beta_2 = \ldots = \beta_n = 0, \quad \text{from Theorem 2 we obtain Corollary 7.} \hspace{1cm} \square\]
Corollary 8. Let $\gamma$, $\beta_j$ be complex numbers, $j = 1, n$, $0 < \Re \gamma \leq 1$ and $f_j$, $g_j \in S$, $f'_j$, $g'_j \in P$, $f_j(z) = z + a_2 z^2 + \ldots$, $g_j(z) = z + b_2 z^2 + \ldots$, $j = 1, n$.

If
\[
\sum_{j=1}^{n} |\beta_j| \leq \frac{\Re \gamma}{4},
\]
then the integral operator $K_n$ defined by (29) belongs to the class $S$.

Proof. We take $\delta = 1$ and $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ in Theorem 2.

Corollary 9. Let $\gamma$, $\delta$, $\alpha_j$ be complex numbers, $c = \Re \gamma > 0$, $\Re \delta \geq \Re \gamma$, $j = 1, n$, $f_j$, $g_j \in S$, $f'_j$, $g'_j \in P$, $f_j(z) = z + a_2 z^2 + \ldots$, $g_j(z) = z + b_2 z^2 + \ldots$, $j = 1, n$.

If
\[
\sum_{j=1}^{n} |\alpha_j| \leq \frac{\Re \gamma}{8}, \text{ for } 0 < \Re \gamma < 1
\]

or
\[
\sum_{j=1}^{n} |\alpha_j| \leq \frac{1}{8}, \text{ for } \Re \gamma \geq 1
\]

then the integral operator $G_n$ defined by (33) is in the class $S$.

Proof. For $\beta_1 = \beta_2 = \cdots = \beta_n = 0$ in Theorem 2 we obtain the Corollary 9.

Corollary 10. Let $\gamma$, $\delta$, $\beta_j$ be complex numbers, $c = \Re \gamma > 0$, $\Re \delta \geq \Re \gamma$, $j = 1, n$ and $f_j$, $g_j \in S$, $f'_j$, $g'_j \in P$, $f_j(z) = z + a_2 z^2 + \ldots$, $g_j(z) = z + b_2 z^2 + \ldots$, $j = 1, n$.

If
\[
\sum_{j=1}^{n} |\beta_j| \leq \frac{\Re \gamma}{4}, \text{ for } 0 < \Re \gamma \leq 1
\]

or
\[
\sum_{j=1}^{n} |\beta_j| \leq \frac{1}{4}, \text{ for } \Re \gamma > 1
\]

then the integral operator $Q_n$ defined by (37) belongs to the class $S$.

Proof. For $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ in Theorem 2 we obtain the Corollary 10.
References


