GAUSS-WEINGARTEN AND FRENET EQUATIONS IN THE THEORY OF THE HOMOGENEOUS LIFT TO THE 2-OSCULATOR BUNDLE OF A FINSLER METRIC

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Abstract

In this article we present a study of the subspaces of the manifold \( \text{Osc}_2^2M \), the total space of the 2-osculator bundle of a real manifold \( M \). We obtain the induced connections of the canonical \( N \)-linear metric connection determined by the homogeneous prolongation of a Finsler metric to the manifold \( \text{Osc}_2^2M \). We present the Gauss-Weingarten equations of the associated 2-osculator submanifold. We construct a Frenet frame and we determine the Frenet equations of a curve from the manifold \( \text{Osc}_2^2M \).

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Key words: nonlinear connection, linear connection, induced linear connection.

1 Indroduction

The Sasaki \( N \)-prolongation \( \mathcal{G} \) to the 2-osculator bundle without the null section \( \widetilde{\text{Osc}}^2_2M = \text{Osc}_2^2M \setminus \{0\} \) of a Finslerian metric \( g_{ab} \) on the real manifold \( M \) given by

\[
\mathcal{G} = g_{ab}(x, y^{(1)}) \, dx^a \otimes dx^b + g_{ab}(x, y^{(1)}) \, \delta y^{(1)a} \otimes \delta y^{(1)b} + g_{ab}(x, y^{(1)}) \, \delta y^{(2)a} \otimes \delta y^{(2)b} \quad (*)
\]

is a Riemannian structure on \( \widetilde{\text{Osc}}^2_2M \), which depends only on the metric \( g_{ab} \).

The tensor \( \mathcal{G} \) is not invariant with respect to the homotheties on the fibres of \( \text{Osc}_2^2M \), because \( \mathcal{G} \) is not homogeneous with respect to the variable \( y^{(1)a} \).

In this paper, we use a new kind of prolongation \( \tilde{\mathcal{G}} \) to \( \widetilde{\text{Osc}}^2_2M \), \( ([7]) \), which depends only on the metric \( g_{ab} \). Thus, \( \tilde{\mathcal{G}} \) determines on the manifold \( \widetilde{\text{Osc}}^2_2M \) a Riemannian structure which is \( 0 \)-homogeneous on the fibres of \( \text{Osc}_2^2M \).

Some geometrical properties of \( \tilde{\mathcal{G}} \) are studied: the canonical \( N \)-linear metric connection, the induced linear connections , Gauss-Weingarten and Frenet equations.

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\section{Preliminaries}

As far we know the general theory of submanifolds (in particular the Finsler submanifolds or the complex Finsler submanifolds) is far from being settled ([9], [3], [10], [11]). In [8] and [9] R. Miron and M. Anastasiei give the theory of subspaces in generalized Lagrange spaces. Also, in [6] and [5] R. Miron presented the theory of subspaces in higher order Finsler and Lagrange spaces respectively.

If $\mathcal{M}$ is an immersed manifold in manifold $\mathcal{M}$, a nonlinear connection on $Osc^2\mathcal{M}$ induce a nonlinear connection $\mathcal{N}$ on $Osc^2\mathcal{M}$.

The $d$-tensor $G$ from (*) is not homogeneous with respect to the variable $y^{(1)a}$. This in an inconvenient from the point of view of analytical mechanics. Moreover, the physical dimensions of the terms of $G$ are not the same. This disavantage was corrected by Gh. Atanasiu. He took a new kind of prolongation $\tilde{G}$ to $\tilde{Osc}^2\mathcal{M}$ of the fundamental tensor of a Finsler space, [1], which depends only on the metric $g_{ab}$. Thus, $\tilde{G}$ determines on the manifold $Osc^2\mathcal{M}$ a Riemannian structure which is 0-homogeneous on the fibres of $Osc^2\mathcal{M}$ and $p$ is a positive constant required by applications in order that the physical dimensions of the terms of $\tilde{G}$ be the same. He proved that there exist metrical N-linear connections with respect to the metric tensor $\tilde{G}$.

We take this canonical $N$-linear metric connection $D$ on the manifold $Osc^2\mathcal{M}$ and obtain the induced tangent and normal connections and the relative covariant derivation in the algebra of $d$-tensor fields.

In this paper we get the Gauss-Weingarten formulae of submanifold $Osc^2\mathcal{M}$ for the homogeneos lift $\tilde{G}$ and we construct a Frenet frame and we determine the Frenet equations of a curve from the manifold $Osc^2\mathcal{M}$.

Let us consider the Finsler space $F^n = (M, F)$ ([9]) with the fundamental function $F : TM = Osc\mathcal{M} \to \mathbb{R}$ and the fundamental tensor $g_{ab}(x, y^{(1)})$ on $Osc\mathcal{M}$, given by

$$g_{ab}(x, y^{(1)}) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^{(1)a} \partial y^{(1)b}}, \quad (2.1)$$

where $g_{ab}(x, y^{(1)})$ is positively defined on $Osc\mathcal{M}$.

The canonical 2-spray of $F^n$ is given by

$$\frac{d^2 x^a}{dt^2} + 2G^a \left( x, \frac{dx}{dt} \right) = 0$$

where

$$G^a = \frac{1}{2} \gamma^a_{bc}(x, y^{(1)}) y^{(1)b} y^{(1)c} \quad (2.2)$$

where $\gamma^a_{bc}(x, y^{(1)})$ are the Christoffels symbols of the metric tensor $g_{ab}(x, y^{(1)})$. The canonical nonlinear connection $N$ of the space $F^n$ has the dual coefficients [5]

$$M^a_{\ b}^{(1)} = \frac{\partial G^a}{\partial y^{(1)b}}, \quad M^a_{\ b}^{(2)} = \frac{1}{2} \left\{ \Gamma^a_{\ bc} + M^a_{\ c} M^c_{\ b} \right\}, \quad (2.3)$$
where $\Gamma = y^{(1)a} \frac{\partial}{\partial x^a} + 2y^{(2)a} \frac{\partial}{\partial y^{(1)a}}$.

We have the next decomposition

$$T_wOsc^2M = N_0(w) \oplus N_1(w) \oplus V_2(w), \forall w \in Osc^2M. \quad (2.4)$$

The adapted basis to (2.4) is given by

$$\{ \delta_{\frac{\partial}{\partial x^a}}, \delta_{\frac{\partial}{\partial y^{(1)a}}}, \frac{\partial}{\partial y^{(2)a}} \}, \quad (a = 1, \ldots, n)$$

and its dual basis is $(dx^a, \delta y^{(1)a}, \delta y^{(2)a})$, where

$$\delta_{\frac{\partial}{\partial x^a}} = \frac{\partial}{\partial x^a} - N^{b}_{(1)a} \frac{\partial}{\partial y^{(1)b}} - N^{b}_{(2)a} \frac{\partial}{\partial y^{(2)b}} \quad (2.5)$$

and

$$\delta y^{(1)a} = dy^{(1)a} + M^{a}_{b} dx^b$$
$$\delta y^{(2)a} = dy^{(2)a} + M^{a}_{b} \delta y^b + M^{a}_{b} \delta y^{(2)b} \quad (2.6)$$

We use the next notations:

$$\delta_a = \frac{\delta}{\delta x^a}, \delta_{1a} = \frac{\delta}{\delta y^{(1)a}}, \delta_{2a} = \frac{\delta}{\delta y^{(2)a}}.$$ 

**Proposition 2.1.** The Lie brackets of the vector fields $\{ \delta_{\frac{\partial}{\partial x^a}}, \delta_{\frac{\partial}{\partial y^{(1)a}}}, \frac{\partial}{\partial y^{(2)a}} \}$ are given by

$$[\delta_b, \delta_c] = R^{a}_{(1)c} \delta_{1a} + R^{a}_{(2)c} \delta_{2a},$$
$$[\delta_b, \delta_{1c}] = B^{a}_{(1)c} \delta_{1a} + B^{a}_{(2)c} \delta_{2a},$$
$$[\delta_b, \delta_{2c}] = B^{a}_{(1)c} \delta_{1a} + B^{a}_{(2)c} \delta_{2a},$$
$$[\delta_{1b}, \delta_{1c}] = R^{a}_{(2)c} \delta_{2a},$$
$$[\delta_{1b}, \delta_{2c}] = B^{a}_{(2)c} \delta_{2a},$$

$$[\delta_{1b}, \delta_{2c}] = B^{a}_{(1)c} \delta_{2a}.$$ 

(2.7)
where

\[
R^a_{\ (01)bc} = \delta_cN^a_1 - \delta_bN^a_1,
\]

\[
R^a_{\ (02)bc} = \delta_cN^a_2 - \delta_bN^a_2 + N^a_f R^f_{\ (01)bc},
\]

\[
B^a_{\ (11)bc} = \delta_1cN^a_1, \quad B^a_{\ (12)bc} = \delta_1cN^a_2 - \delta_bN^a_1 + N^a_f B^f_{\ (11)bc},
\]

\[
B^a_{\ (21)bc} = \dot{\delta}_2cN^a_1, \quad B^a_{\ (22)bc} = \dot{\delta}_2cN^a_2 + N^a_f B^f_{\ (21)bc},
\]

\[
R^a_{\ (12)bc} = \delta_1cN^a_1 - \delta_1bN^a_1.
\]

The fundamental tensor $g_{ab}$ determines on the manifold $\tilde{Osc}^2M$ the homogeneous tensor field $\mathcal{G}$, [1],

\[
\mathcal{G} = g_{ab} (x, y^{(1)}) \, dx^a \otimes dx^b + g_{ab} (x, y^{(1)}) \, \delta y^{(1)a} \otimes \delta y^{(1)b} +
\]

\[
+ g_{ab} (x, y^{(1)}) \, \delta y^{(2)a} \otimes \delta y^{(2)b},
\]

where

\[
g_{ab} (x, y^{(1)}) = \frac{p^2}{\|y^{(1)}\|^2} g_{ab} (x, y^{(1)}),
\]

\[
g_{ab} (x, y^{(2)}) = \frac{p^4}{\|y^{(1)}\|^2} g_{ab} (x, y^{(1)}),
\]

\[
\|y^{(1)}\|^2 = g_{ab} y^{(1)a} y^{(1)b}.
\]

This is homogeneous tensor field with respect to $y^{(1)a}$, $y^{(2)a}$ and $p$ is a positive constant required by applications in order that the physical dimensions of the terms of $\mathcal{G}$ be the same.

Let $\tilde{M}$ be a real, m-dimensional manifold, immersed in $M$ through the immersion $i : \tilde{M} \rightarrow M$. Locally, $i$ can be given in the form

\[
x^a = x^a (u^1, ..., u^m), \quad \text{rank} \| \frac{\partial x^a}{\partial u^\alpha} \| = m.
\]

The indices $a, b, c, ...$ run over the set $\{1, ..., n\}$ and $\alpha, \beta, \gamma, ...$ run on the set $\{1, ..., m\}$. We assume $1 \leq m < n$. We take the immersed submanifold $\tilde{Osc}^2\tilde{M}$ of the manifold $Osc^2M$, by the immersion $\tilde{Osc}^2i : Osc^2M \rightarrow Osc^2M$. The parametric equations of the submanifold $Osc^2M$ are
Gauss-Weingarten and Frenet equations in the theory of the homogeneous lift.

\[
\begin{align*}
\begin{cases}
x^a = x^a (u^1, \ldots, u^m), \, \text{rank } \| \frac{\partial x^a}{\partial u^\alpha} \| = m \\
y^{(1)a} = \frac{\partial x^a}{\partial u^\alpha} v^{(1)\alpha} \\
2 y^{(2)a} = \frac{\partial y^{(1)a}}{\partial u^\alpha} v^{(1)\alpha} + 2 \frac{\partial y^{(1)a}}{\partial v^{(1)\alpha}} v^{(2)\alpha},
\end{cases}
\end{align*}
\]

where

\[
\begin{cases}
\frac{\partial x^a}{\partial u^\alpha} = \frac{\partial y^{(1)a}}{\partial u^\alpha} = \frac{\partial y^{(2)a}}{\partial u^\alpha} \\
\frac{\partial y^{(1)a}}{\partial u^\alpha} = \frac{\partial y^{(2)a}}{\partial u^\alpha}.
\end{cases}
\]

The restriction of the fundamental function $F$ to the submanifold $\tilde{Osc} \tilde{M}$ is

\[\tilde{F} (u, v^{(1)}) = F \left( x (u), y \left( u, v^{(1)}, v^{(2)} \right) \right)\]

and we call $\tilde{F}^m = (\tilde{M}, \tilde{F})$ the induced Finsler subspaces of $F^n$ and $\tilde{F}$ the induced fundamental function.

Let $B^a_\alpha (u) = \frac{\partial x^a}{\partial u^\alpha}$ and $g_{\alpha \beta}$ the induced fundamental tensor,

\[g_{\alpha \beta} (u, v^{(1)}) = g_{ab} \left( x (u), y \left( u, v^{(1)} \right) \right) B^a_\alpha B^b_\beta.\]

We obtain a system of d-vectors \{B^a_\alpha, B^a_\bar{\alpha}\} which determines a moving frame $\mathcal{R} = \{(u, v^{(1)}, v^{(2)}); B^a_\alpha (u), B^a_{\bar{\alpha}} (u, v^{(1)}, v^{(2)})\}$ in $Osc^2 \tilde{M}$ along the submanifold $Osc^2 \tilde{M}$.

Its dual frame will be denoted by $\mathcal{R}^* = \{B^a_\alpha (u, v^{(1)}, v^{(2)}), B^a_{\bar{\alpha}} (u, v^{(1)}, v^{(2)})\}$. This is also defined on an open set $\pi^{-1} (\tilde{U}) \subset Osc^2 \tilde{M}$, $\tilde{U}$ being a domain of a local chart on the submanifold $\tilde{M}$.

The conditions of duality are given by:

\[B^a_\beta B^\alpha_a = \delta^\alpha_\beta, \quad B^a_\beta B^\bar{\alpha}_a = 0, \quad B^a_\alpha B^\alpha_a = 0, \quad B^\alpha_a B^\beta_a = \delta^\alpha_\beta\]

\[B^a_\alpha B^a_\beta + B^a_{\bar{\alpha}} B^a_{\bar{\beta}} = \delta^a_b.\]

The restriction of the of the nonlinear connection $N$ to $Osc^2 \tilde{M}$ uniquely determines an induced nonlinear connection $\tilde{N}$ on $Osc^2 \tilde{M}$ with the dual coefficients ([2],[13])

\[
\begin{align*}
\tilde{M}^{\alpha}_{\beta} &= B^a_{\alpha} \left( B^a_{\beta} + M^a_b B^b_{\beta} \right), \\
\tilde{M}^{\alpha}_{\bar{\beta}} &= B^a_{\alpha} \left( \frac{\partial B^a_{\beta}}{\partial u^\alpha} v^{(1)\delta} v^{(1)\gamma} + B^a_{\delta \beta} v^{(2)\delta} + M^a_b B^b_{\beta} + M^a_{\beta b} B^b_{\beta} \right),
\end{align*}
\]

where $M^a_1 b$, $M^a_2 b$ are the dual coefficients of the $N$. 

The cobasis \((dx^i, \delta y^{(1)a}, \delta y^{(2)a})\) restricted to \(Osc^2M\) is uniquely represented in the moving frame \(R\) in the following form ([2], [12]):

\[
\begin{cases}
   dx^a = B^a_{\beta} du^\beta \\
   \delta y^{(1)a} = B^a_\alpha \delta v^{(1)\alpha} + B^a_\beta K^\beta_{(1)\alpha} du^\beta \\
   \delta y^{(2)a} = B^a_\alpha \delta v^{(2)\alpha} + B^a_\beta K^\beta_{(2)\alpha} du^\alpha 
\end{cases}
\]

(2.13)

where

\[
K^\alpha_{(1)\beta} = B^a_\alpha \left( B^a_{0\beta} + M^a_{(1)\beta} B^b_{_{(1)b}} \right)
\]

\[
K^\alpha_{(2)\beta} = B^a_\alpha \left( \frac{1}{2} \partial B^a_{\gamma\delta} v^{(1)\gamma} + B^a_{\delta\beta} v^{(2)\delta} + M^a_{(1)\beta} B^b_{(1)\beta} + M^a_{(2)\beta} B^b_{(2)\beta} - B^a_{\gamma} B_{d}^{\gamma} \left( M^a_{(1)\beta} B^b_{(1)\beta} \right) \right)
\]

(2.14)

are mixed d-tensor fields.

A linear connection \(D\) on the manifold \(Osc^2M\) is called \textbf{metrical N-linear connection} with respect to \(\hat{G}\), if \(D\hat{G} = 0\) and \(D\) preserves by parallelism the distributions \(N_0, N_1\) and \(V_2\). The coefficients of the N-linear connections \(D\Gamma (N)\) will be denoted with \(\left( L^a_{(i)bc} \right), \left( V^a_{(i)bc} \right)\), \((i = 0, 1, 2)\). \(\) \(\)

**Theorem 2.2.** ([1]) There exist metrical \(N\)-linear connections \(D\Gamma (N)\) on \(Osc^2M\), with respect to the homogeneous prolongation \(\hat{G}\), which depend only on the metric \(g_{ab}(x, y^{(1)})\). One of these connections has

the ”horizontal” coefficients

\[
H^a_{(0)bc} = \frac{1}{2} g^{ad} \left( \delta_b g_{cd} + \delta_c g_{bd} - \delta_d g_{bc} \right)
\]

\[
L^a_{(10)bc} = \frac{1}{2} g^{ad} \left( \delta_b g_{cd} + \delta_c g_{bd} - \delta_d g_{bc} \right)_{(1)}
\]

(2.15)

\[
V^a_{(10)bc} = \frac{1}{2} g^{ad} \left( \delta_b g_{cd} + \delta_c g_{bd} - \delta_d g_{bc} \right)_{(1)}
\]

\[
V^a_{(20)bc} = \frac{1}{2} g^{ad} \left( \delta_b g_{cd} + \delta_c g_{bd} - \delta_d g_{bc} \right)_{(2)}
\]
Gauss-Weingarten and Frenet equations in the theory of the homogeneous lift.

the "1-vertical" coefficients

\[
\frac{H}{(01)}_{bc} = \frac{1}{2} g^{ad} (\delta_{db} g_{cd} + \delta_{dc} g_{bd} - \delta_{1d} g_{bc})
\]
\[
\frac{V_1}{(11)}_{bc} = \frac{1}{2} g^{ad} \begin{pmatrix}
\delta_{db} g_{(1)cd} + \delta_{dc} g_{bd} - \delta_{1d} g_{(1)bc} \\
\delta_{db} g_{(2)cd} + \delta_{dc} g_{bd} - \delta_{1d} g_{(2)bc}
\end{pmatrix}
\]

and the "2-vertical" coefficients

\[
\frac{H}{(02)}_{bc} = \frac{V_1}{(12)}_{bc} = \frac{V_2}{(22)}_{bc} = 0.
\]

It is called the canonical N-linear metric connection.

This linear connection will be used throughout this paper.

For this N-linear connection, we have the operators \( V_i \) \((i = 0, 1, 2; V_0 = H)\) which are given by the following relations

\[
V_i DX^a = dX^a + V_i^a X^b, \forall X \in \mathcal{F}\left(\tilde{Osc}^2 M\right),
\]

where

\[
\frac{H}{(00)}_{bc} = \frac{H}{(01)}_{bc} \frac{\delta y(1)}{c} + \frac{H}{(02)}_{bc} \frac{\delta y(2)}{c}
\]
\[
\frac{V_1}{(10)}_{bc} = \frac{V_1}{(11)}_{bc} \frac{\delta y(1)}{c} + \frac{V_1}{(12)}_{bc} \frac{\delta y(2)}{c}
\]
\[
\frac{V_2}{(20)}_{bc} = \frac{V_2}{(21)}_{bc} \frac{\delta y(1)}{c} + \frac{V_2}{(22)}_{bc} \frac{\delta y(2)}{c}.
\]

We call these operators the horizontal, 1- and 2-vertical covariant differentials. The 1-forms \( H^a_b, V_1^a_b, V_2^a_b \) will be called the horizontal, 1- and 2-vertical 1-form. From (2.17) we get that the horizontal, 1- and 2-vertical 1-form are

\[
\frac{H}{(00)}_{bc} = \frac{H}{(01)}_{bc} \frac{\delta y(1)}{c} + \frac{H}{(02)}_{bc} \frac{\delta y(2)}{c}
\]
\[
\frac{V_1}{(10)}_{bc} = \frac{V_1}{(11)}_{bc} \frac{\delta y(1)}{c} + \frac{V_1}{(12)}_{bc} \frac{\delta y(2)}{c}
\]
\[
\frac{V_2}{(20)}_{bc} = \frac{V_2}{(21)}_{bc} \frac{\delta y(1)}{c} + \frac{V_2}{(22)}_{bc} \frac{\delta y(2)}{c}.
\]
3 The relative covariant derivatives

Let $D\Gamma(N)$, the canonical $N$-linear metric connection of the manifold $Osc^2 M$. A classical method to determine the laws of derivation on a Finsler submanifold is the type of the coupling([5],[6],[8],[9]).

**Definition 3.1.** We call a coupling of the canonical $N$-linear metric connection $D$ to the induced nonlinear connection $\tilde{N}$ along $\tilde{\text{Osc}}^2 M$ the operators $V_i$ defined by the operators $D_i(i = 0, 1, 2; V_0 = H) \ (2.18)$ with the property

$$V_i DX^a = V_i DX^a, \ (i = 0, 1, 2; V_0 = H) \text{ (modulo 2.13)} \ (3.1)$$

Here

$$V_i DX^a = dX^a + V_i \omega^a_b X^b, \ \forall X \in \mathcal{F}(\tilde{Osc}^2 M). \ (3.2)$$

The 1-forms $\tilde{\omega}^a_{(i)} b, (i = 0, 1, 2)$ are the connection 1-forms of the coupling $\tilde{D}$.

**Theorem 3.2.** The coupling of the $N$-linear connection $D$ to the induced nonlinear connection $\tilde{N}$ along $\tilde{\text{Osc}}^2 M$ is locally given by the set of coefficients $\tilde{D}\Gamma(N) = \left( \begin{array}{c} V_i \\ \tilde{L}^a_{(i0)} \tilde{b} \\ \tilde{C}^a_{(i1)} b \\ \tilde{C}^a_{(i2)} b \end{array} \right)$, $(i = 0, 1, 2; V_0 = H)$ where

$$V_i \tilde{L}^a_{(i0)} b = \tilde{L}^a_{(i0)} b + \tilde{C}^a_{(i1)} b \delta^d K^\delta_{(1)}$$

$$V_i \tilde{C}^a_{(i1)} b = \tilde{C}^a_{(i1)} b$$

$$V_i \tilde{C}^a_{(i2)} b = 0, (i = 0, 1, 2; V_0 = H). \ (3.3)$$

**Proof.** From (3.1), (3.2), (2.18), and (2.13) we obtain

$$V_i \tilde{L}^a_{(i0)} b = \tilde{L}^a_{(i0)} b + \tilde{C}^a_{(i1)} b \delta^d K^\delta_{(1)} + \tilde{C}^a_{(i2)} b \delta^d K^\delta_{(2)}$$

$$V_i \tilde{C}^a_{(i1)} b = \tilde{C}^a_{(i1)} b + \tilde{C}^a_{(i2)} b \delta^d K^\delta_{(1)}$$

$$V_i \tilde{C}^a_{(i2)} b = \tilde{C}^a_{(i2)} b, (i = 0, 1, 2; V_0 = H).$$

and from (2.17) we get (3.3).
Definition 3.3. We call the **induced tangent connection** on $\tilde{\text{Osc}}^2 \tilde{M}$ by the canonical $N$-linear metric connection $D$, the couple of the operators $D^\perp$, $(i = 0, 1, 2; V_0 = H)$ which are defined by

$$D^\perp X^\alpha = B^\alpha_d DX^b, \quad \text{for } X^a = B^a_\gamma X^\gamma$$

(3.4)

where

$$D^\perp X^\alpha = dX^\alpha + X^\beta V^\alpha_{\beta \gamma},$$

(3.5)

and $V^\alpha_{\beta \gamma}$, $(i = 0, 1, 2; V_0 = H)$ are called the **tangent connection 1-forms**.

We have

**Theorem 3.4.** The tangent connections 1-forms are as follows:

$$V^\alpha_{\beta \gamma} = V^\alpha_{(i0)} \delta_{\beta \delta} \delta_{\gamma \delta} + V^\alpha_{(i1)} \delta_{\beta \delta} \delta_{\gamma \delta} + V^\alpha_{(i2)} \delta_{\beta \delta} \delta_{\gamma \delta},$$

(3.6)

where

$$V^\alpha_{(i0)} \delta_{\beta \delta} = B^\alpha_d \left( B^d_\beta + B^d_\beta L^d_\beta \delta_{\gamma \delta} \right),$$

$$V^\alpha_{(i1)} \delta_{\beta \delta} = B^\alpha_d B^d_\beta \tilde{C}^d_\beta \delta_{\gamma \delta},$$

$$V^\alpha_{(i2)} \delta_{\beta \delta} = 0, (i = 0, 1, 2; V_0 = H).$$

**Proof.** From (3.2), (3.5) and (3.4) we have

$$V^\alpha_{(i0)} \delta_{\beta \delta} = B^\alpha_d \left( B^d_\beta + B^d_\beta L^d_\beta \delta_{\gamma \delta} \right),$$

$$V^\alpha_{(i1)} \delta_{\beta \delta} = B^\alpha_d B^d_\beta \tilde{C}^d_\beta \delta_{\gamma \delta},$$

$$V^\alpha_{(i2)} \delta_{\beta \delta} = B^\alpha_d B^d_\beta \tilde{C}^d_\beta \delta_{\gamma \delta}, (i = 0, 1, 2; V_0 = H).$$

and from (2.17) we get (3.7). 


Definition 3.5. We call the **induced normal connection** on $\tilde{\text{Osc}}^2 \tilde{M}$ by the canonical $N$-linear metric connection $D$, the couple of the operators $D^\perp$, $(i = 0, 1, 2; V_0 = H)$ which are defined by

$$D^\perp X^\alpha = B^\alpha_d DX^b, \quad \text{for } X^a = B^a_\gamma X^\gamma$$

(3.8)
where
\[ V_i D^\bot X^\beta = dX^\beta + X^\beta V_i \]
and \( V_i \), \((i = 0, 1, 2; V_0 = H)\) are called the normal connection 1-forms.

We have

**Theorem 3.6.** The normal connections 1-forms are as follows:

\[ V_i \omega_{ij} = L_i^{(0)} du^\delta + C_i^{(1)} \delta v^{(1)} + C_i^{(2)} \delta v^{(2)} \]

where

\[ L_i^{(0)} = B_{i}^{\delta \alpha} \left( \frac{\partial B_{\beta}^{\delta}}{\partial u^\alpha} + B_{\beta}^{f} L_i^{(0)} f_\delta \right) \]

\[ C_i^{(1)} = B_{i}^{\delta \alpha} \left( \frac{\partial B_{\beta}^{\delta}}{\partial u^\alpha} + B_{\beta}^{f} C_i^{(1)} f_\delta \right) \]

\[ C_i^{(2)} = 0, \quad (i = 0, 1, 2; V_0 = H) \]

**Proof.** From (3.2),(3.8),(3.9) and (2.13) we obtain

\[ L_i^{(0)} = B_{i}^{\delta \alpha} \left( \frac{\partial B_{\beta}^{\delta}}{\partial u^\alpha} + B_{\beta}^{f} L_i^{(0)} f_\delta \right) \]

\[ C_i^{(1)} = B_{i}^{\delta \alpha} \left( \frac{\partial B_{\beta}^{\delta}}{\partial u^\alpha} + B_{\beta}^{f} C_i^{(1)} f_\delta \right) \]

\[ C_i^{(2)} = 0, \quad (i = 0, 1, 2; V_0 = H) \]

Now, we can define the relative (or mixed) covariant derivatives \( V_i \), \((i = 0, 1, 2; V_0 = H)\).

**Theorem 3.7.** The relative covariant (mixed) derivatives in the algebra of mixed d-tensor fields are the operators \( V_i \), \((i = 0, 1, 2; V_0 = H)\) for which the following properties hold:

\[ V_i \nabla f = df, \quad \forall f \in \mathcal{F}(Osc^2 M) \]

\[ V_i \nabla X^\alpha = D^\alpha, \quad V_i \nabla X^\alpha = D^\bot X^\alpha, \quad V_i \nabla X^\alpha = D^\bot X^\alpha, \quad (i = 0, 1, 2; V_0 = H) \]
$V_i$, $\omega^\alpha_b$, $\omega^\beta_\alpha$, $\omega^\pi_\beta$ are called the connection 1-forms of $\nabla_i, (i = 0, 1, 2; V_0 = H)$.

4 The Gauss-Weingarten formulae

In the theory of the submanifolds we are interested in finding the moving equations of the moving frame $R$ along $Osc^2 M$.

These equations, called also Gauss-Weingarten formulae, are obtained when the relative covariant derivatives of the vector fields from $R$ are expressed again in the frame $R$.

Thus we have

Theorem 4.1. The following Gauss-Weingarten formulae hold:

$$\nabla_i B^a_\alpha = B^a_\delta \Pi^\delta_\alpha,$$

$$\nabla_i B'^a_\alpha = -B'^a_\delta \Pi'^\delta_\alpha,$$

where

$$\Pi^\delta_\alpha = H^{(0)}_{\alpha} \delta^\delta_\beta d^\beta + H^{(1)}_{\alpha} \delta^\delta_\beta \delta v^{(1)\beta} + H^{(2)}_{\alpha} \delta^\delta_\beta \delta v^{(2)\beta},$$

$$\Pi'^\delta_\alpha = g^{\alpha\sigma} \delta^\delta_\beta \Pi'^\sigma_\alpha,$$

and the d-tensors

$$H^{(0)}_{\alpha} \delta^\delta_\beta = B^d_\alpha \left( B^d_\alpha + B^f_\alpha L^d_{(10)f\beta} \right),$$

$$H^{(1)}_{\alpha} \delta^\delta_\beta = B^f_\alpha C^d_{(1) f\beta},$$

$$H^{(2)}_{\alpha} \delta^\delta_\beta = B^f_\alpha C^d_{(2) f\beta},$$

are the fundamental d-tensors of the second order of manifold $Osc^2 M$, $(i = 0, 1, 2, V_0 = H)$. 


Proof. From (2.15), (2.16) and (2.17) we have

\[ H \nabla B^a_\alpha = B^a_{\alpha | \beta \delta} du^\beta + B^a_\alpha |_{0 \beta} \delta V^{(1)\delta} + B^a_\alpha |_{0 \beta} \delta V^{(2)\delta} \]

\[ = \left( \frac{\delta B^a_\alpha}{\delta u^\beta} + \frac{H}{(00)_{\beta \delta}} B^b_\beta - \frac{H}{(00)_{\alpha \beta}} B^a_\delta \right) du^\beta + \]

\[ + \left( \frac{\delta B^a_\alpha}{\delta v^{(1)\beta}} + \frac{C_{(p1)_{\beta \delta}}^a}{\alpha \beta} B^b_\alpha - \frac{H}{(01)_{\alpha \beta}} B^a_\delta \right) \delta V^{(1)\beta} + \]

\[ + \left( \frac{\delta B^a_\alpha}{\delta v^{(2)\beta}} + \frac{\delta C_{(p2)_{\beta \delta}}^a}{\alpha \beta} B^b_\alpha - \frac{H}{(02)_{\alpha \beta}} B^a_\delta \right) \delta V^{(2)\beta} \]

\[ = B^a_{\alpha \beta} du^\beta + B^b_\alpha \left( \frac{H}{(00)_{\beta \delta}} du^\beta + \frac{C_{(01)_{\beta \delta}}^a}{\alpha \beta} \delta V^{(1)\beta} + \frac{H}{(02)_{\beta \delta}} \delta V^{(2)\beta} \right) - \]

\[ - B^a_\delta \left[ B^d_\beta \left( B^d_{\alpha \beta} + B^d_\alpha \frac{H}{(00)_{f \delta}} \right) du^\beta + B^d_\beta B^d_{\alpha \beta} \frac{H}{(01)_{f \beta}} \delta V^{(1)\beta} + \right] \]

\[ + B^d_\beta B^d_\alpha \frac{H}{(02)_{f \beta}} \delta V^{(2)\beta} \].

Using (4.3) we get the relation (4.1) for \( V_0 = H \).

Now, by applying \( H \nabla \) to the both sides of the equations \( g_{ab} B^a_\alpha B^b_\beta = 0 \) one get

\[ g_{ab} B^a_\delta B^b_\beta + g_{ab} B^a_\alpha H B^b_\beta = 0. \]

Multiplying these relation with \( B^a_\alpha \) we obtain

\[ g_{bd} H \nabla B^b_\beta - B^a_\delta B^b_\gamma H \nabla B^b_\beta = -B^a_\delta B^\delta_{\beta \gamma} H. \]

But \( B^a_\delta B^b_\delta g_{ab} \nabla B^b_\beta = 0 \). Consequently, we obtain the relations (4.2) for \( V_0 = H \).

Analogously, for the operators \( V_i, (i = 1, 2) \) one gets the other relations.

\( \square \)

5 Curves in the manifold \( \text{Osc}^2 M \)

In this section we construct a Frenet frame and determine the Frenet equations for a curve in the manifold \( \text{Osc}^2 M \).

The start point of these researchs is the Bejancu and Farran results in case of vertical bundle of \( TM \) ([3]). We construct a Frenet frame and derive all the Frenet equations for a
curve in the manifold \( \widetilde{\text{Osc}^2 M} \). This enables us to state a fundamental theorem for curves in manifold \( \widetilde{\text{Osc}^2 M} \).

Let \( c : t \to (x^a(t)) \) a smooth curve in \( M \), \( t \) a real parameter and \( s(t) \) a parameter change. On the manifold \( \text{Osc}^2 M \) with the local coordinates \((x^a, y^{(1)a}, y^{(2)a})\), the curve \( c \) induce a curve \( C \) with the property

\[
(x^a(t), y^{(1)a}(t)) = \frac{dx^a}{dt}, \quad y^{(2)a}(t) = \frac{1}{2} \frac{d^2 x^a}{dt^2}.
\]

If we change the parameter on \( C \), we have

\[
y^{(1)a}(s) = \frac{ds}{dt} \frac{dx^a}{ds} = v^{(1)} \frac{dx^a}{ds},
\]

\[
y^{(2)a}(s) = \frac{1}{2} \frac{d}{dt} (v^{(1)} \frac{dx^a}{ds}) = \frac{1}{2} \left( \frac{d^2 s}{dt^2} \frac{dx^a}{ds} + \left( v^{(1)} \right)^2 \frac{d^2 x^a}{ds^2} \right)\]

\[(5.1)\]

where we noted

\[
v^{(1)} = \frac{ds}{dt}, \quad v^{(2)} = \frac{d^2 s}{dt^2}.
\]

Thus, on \( C \), we may consider the parameters \((s, v^{(1)}, v^{(2)})\), and from the above calculus we get the parametric equations in \( s \):

\[
\begin{align*}
x^a &= x^a(s), \\
y^{(1)a} &= v^{(1)} \frac{dx^a}{ds}, \\
2y^{(2)a} &= v^{(2)} \frac{dx^a}{ds} + \left( v^{(1)} \right)^2 \frac{d^2 x^a}{ds^2}.
\end{align*}
\]

We use notation \( x'^a = \frac{dx^a}{ds} \) and \( x''^a = \frac{d^2 x^a}{ds^2} \), thus the above equations become:

\[
\begin{align*}
x^a &= x^a(s), \\
y^{(1)a} &= v^{(1)} x'^a, \\
2y^{(2)a} &= v^{(2)} x'^a + \left( v^{(1)} \right)^2 x''^a.
\end{align*}
\]

The tangent vectors along \( C \) are \( \left\{ \frac{\partial}{\partial s}, \frac{\partial}{\partial v^{(1)}}, \frac{\partial}{\partial v^{(2)}} \right\} \), and their connection with the tangent vectors \( \left\{ \frac{\partial}{\partial x'^a}, \frac{\partial}{\partial y^{(1)a}}, \frac{\partial}{\partial y^{(2)a}} \right\} \) is obtained by using the Jacobi matrix of these transformations

\[
\frac{\partial}{\partial s} = x'^a \frac{\partial}{\partial x'^a} + v^{(1)} x'^a \frac{\partial}{\partial y^{(1)a}} + \left( v^{(1)^2} x'^a + v^{(2)} x'^a \right) \frac{\partial}{\partial y^{(2)a}},
\]

\[
\frac{\partial}{\partial v^{(1)}} = x'^a \frac{\partial}{\partial y^{(1)a}} + v^{(1)} x'^a \frac{\partial}{\partial y^{(2)a}};
\]

\[
\frac{\partial}{\partial v^{(2)}} = x'^a \frac{\partial}{\partial y^{(2)a}};
\]
Let \((M,F)\) a Finsler space and \(g_{ab}(x,y^{(1)}) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^a \partial y^b}\) the fundamental tensor and 
\(s(t) = \int_0^t F(x(t),y^{(1)}(t)) \, dt\) the arc length parameter on \(C\). We have \(\frac{ds}{dt} = F(x(t),y^{(1)}(t))\) along of curve \(C\). A consequence of homogeneity of \(F\) it follows that \(v^{(1)} = \frac{ds}{dt} = F(x(t),v^{(1)}(t)) = F(x(s),v^{(1)}(s)) = \frac{v^{(1)}}{F}(x(s),y^{(1)}(s))\). We deduce that \(F(x(s),y^{(1)}(s)) = 1\), which is true on \(C\) restrictioned to the manifold \(Osc^1 M \equiv TM\).

Let \(G\) the Sasaki prolongation of the fundamental tensor \(g_{ab}\) to the manifold \(Osc^2 M\), (*)

\[
G = g_{ab}(x,y^{(1)}) \, dx^a \otimes dx^b + g_{ab}(x,y^{(1)}) \, \delta y^{(1)a} \otimes \delta y^{(1)b} + 
+ g_{ab}(x,y^{(1)}) \, \delta y^{(2)a} \otimes \delta y^{(2)b}
\]

and \(\nabla_i\), the canonical \(N\)-linear metric connection on the manifold \(Osc^2 M\), with the coefficients

\[
L_{(i0)}^{a}{}_{bc} = \frac{1}{2} g^{ad} \left( \delta_b g_{cd} + \delta_c g_{bd} - \delta_d g_{bc} \right)
\]

\[
C_{(i1)}^{a}{}_{bc} = \frac{1}{2} g^{ad} \left( \delta_{1b} g_{cd} + \delta_{1c} g_{bd} - \delta_{1d} g_{bc} \right) \quad (i = 0, 1, 2) \quad (5.2)
\]

\[
C_{(i2)}^{a}{}_{bc} = 0,
\]

and \(\nabla_i\), \((i = 0, 1, 2)\) the covariant derivatives operators.

Let

\[
g_{ab}^{a}{}_{c} = \frac{1}{2} g^{ad} \frac{\partial g_{bd}}{\partial y^{(1)c}} \quad (5.3)
\]

and \(\nabla_i\), the \(N\)-linear connection of Berwald type with the coefficients

\[
\begin{pmatrix}
L_{(00)}^{b}{}_{bc}, & L_{(10)}^{b}{}_{bc}, & L_{(20)}^{b}{}_{bc}, & C_{(01)}^{b}{}_{bc}, & C_{(11)}^{b}{}_{bc}, & C_{(21)}^{b}{}_{bc}, & C_{(02)}^{b}{}_{bc}, & C_{(12)}^{b}{}_{bc}, & C_{(22)}^{b}{}_{bc}
\end{pmatrix} \quad (5.4)
\]

where

\[
L_{(00)}^{b}{}_{bc} = L_{(00)}^{a}{}_{bc}, \quad L_{(10)}^{b}{}_{bc} = L_{(10)}^{a}{}_{bc}, \quad L_{(20)}^{b}{}_{bc} = L_{(20)}^{a}{}_{bc}, \quad C_{(01)}^{b}{}_{bc} = C_{(01)}^{a}{}_{bc}, \quad C_{(11)}^{b}{}_{bc} = C_{(11)}^{a}{}_{bc}, \quad C_{(21)}^{b}{}_{bc} = C_{(21)}^{a}{}_{bc}, \\
C_{(02)}^{b}{}_{bc} = C_{(02)}^{a}{}_{bc}, \quad C_{(12)}^{b}{}_{bc} = C_{(12)}^{a}{}_{bc}, \quad C_{(22)}^{b}{}_{bc} = C_{(22)}^{a}{}_{bc}, \\
B_{(00)}^{b}{}_{bc} = B_{(00)}^{a}{}_{bc}, \quad B_{(10)}^{b}{}_{bc} = B_{(10)}^{a}{}_{bc}, \quad B_{(20)}^{b}{}_{bc} = B_{(20)}^{a}{}_{bc},
\]

\[
\begin{pmatrix}
C_{(01)}^{a}{}_{bc} = 0, & C_{(11)}^{a}{}_{bc}, & C_{(21)}^{a}{}_{bc} = 0, \\
C_{(02)}^{a}{}_{bc} = 0, & C_{(12)}^{a}{}_{bc} = 0, & C_{(22)}^{a}{}_{bc} = 0
\end{pmatrix} \quad (5.5)
\]
where \( L_{(00)}^{a} \), \( C_{(11)}^{a} \), \( C_{(22)}^{a} \), and \( B_{(11)}^{a} \), \( B_{(22)}^{a} \) are given by the formulas (5.2) and (2.8),

\[
B_{(11)}^{a} = \delta_{1c}N_{1}^{a}b, \quad B_{(12)}^{a} = \delta_{1c}N_{2}^{a}b - \delta_{b}N_{1}^{a}c + N_{1}^{a}f, \quad B_{(21)}^{a} = \delta_{1c}N_{1}^{a}b - \delta_{1b}N_{1}^{a}c.
\]

Then, we have

\[
\mathbb{G} \left( \frac{\partial}{\partial v^{(2)}}, \frac{\partial}{\partial v^{(2)}} \right) = g_{ab} \left( x(t), \dot{x}(t) \right) \frac{dx^{a}}{dt} \frac{dx^{b}}{dt}.
\]  

(5.6)

If we take the parameter \( s \), it follows that

\[
\mathbb{G} \left( \frac{\partial}{\partial v^{(2)}}, \frac{\partial}{\partial v^{(2)}} \right) = g_{ab}x^{a}x^{b} = F^{2}(x(s), y^{(1)}(s)) = 1.
\]  

(5.7)

We study geometric objects along curve \( C \) in points \( (x^{a}(s), y^{(1)a}(s), y^{(2)a}(s)) \) where \( v^{(1)} \neq 0 \).

An other consequence of homogeneity of \( F \) is that \( g_{ab}, g^{ab} \) and \( L_{(00)}^{a} \) (2.15) (or \( L_{(00)}^{a} \), \( i = 0, 1, 2 \) from (5.2)) are positive homogeneous of degree zero, while \( G^{a}, G_{b}^{a} \) and \( g_{b}^{a}c \), (5.3), are positive homogeneous of degrees 2, 1 and -1. The functions \( G^{a} \) (2.2) are

\[
G^{a}(x, y^{(1)}) = \frac{1}{4}g^{ab}(x, y^{(1)}) \left( \frac{\partial^{2} F^{*}}{\partial y^{(1)b}} \partial x^{c}y^{(1)c} - \frac{\partial F^{*}}{\partial x^{b}} \right)(x, y^{(1)}), \quad F^{*} = F^{2}.
\]

We deduce:

\[
g_{ab}(x(s), v^{(1)}x^{c}(s)) = g_{ab}(x(s), x^{c}(s)), 
\]  

(5.8)

\[
g^{ab}(x(s), v^{(1)}x^{c}(s)) = g^{ab}(x(s), x^{c}(s)),
\]

\[
x^{b}(s) L_{(00)}^{a} \left( x(s), v^{(1)}x^{c}(s) \right) = x^{b}(s) L_{(00)}^{a} \left( x(s), x^{c}(s) \right) = G_{b}^{a}(x(s), x^{c}(s)),
\]  

(5.9)

\[
x^{b}(s) G_{b}^{a} \left( x(s), v^{(1)}x^{c}(s) \right) = v^{(1)}x^{b}(s) G_{b}^{a} \left( x(s), x^{c}(s) \right) = 2v^{(1)}G^{a}(x(s), x^{c}(s)),
\]

(5.10)

\[
x^{b}(s) g_{b}^{a}c(x(s), x^{c}(s)) = 0,
\]

(5.11)

\[
g_{b}^{a}c(x(s), v^{(1)}x^{c}(s)) = \frac{1}{v^{(1)}} g_{b}^{a}c(x(s), x^{c}(s)).
\]

**Remark 5.1.** Here and in the sequel we use the vector notations \( x(s) \) and \( x^{c}(s) \) to represent the vectors \( (x^{0}(s), ..., x^{m}(s)) \) and \( (x^{0}(s), ..., x^{m}(s)) \), respectively. Also, the components of a geometric object \( T_{def...}^{abc...} \) at the point \( (x(s), x^{c}(s)) \) we denote them by \( T_{def...}^{abc...}(s) \).
We say that a vector field \( X \in \mathcal{X} \left( \tilde{Osc}^2 M \right) \) along \( Osc^2 \mathcal{C} \) is **projectable on** \( \mathcal{C} \) if locally at any point \( (x(s), v^{(1)}x'(s), y^{(2)a}(s)) \in \tilde{Osc}^2 \mathcal{C} \) we have

\[
X \left( x(s), v^{(1)}x'(s), y^{(2)a}(s) \right) = X^{(0)a}(s) \frac{\delta}{\delta x^a} \left( x(s), v^{(1)}x'(s), y^{(2)a}(s) \right) + X^{(1)a}(s) \frac{\delta}{\delta y^{(1)a}} \left( x(s), v^{(1)}x'(s), y^{(2)a}(s) \right) + X^{(2)a}(s) \frac{\partial}{\partial y^{(2)a}} \left( x(s), v^{(1)}x'(s), y^{(2)a}(s) \right),
\]

or, equivalently, the local components of \( X \) at any point of \( \mathcal{C} \) depend only on the arc length parameter \( s \) of \( \mathcal{C} \). The above name is justified because \( X \) given by (5) on \( Osc^2 \mathcal{C} \) defines a vector field \( X^* \) on \( \mathcal{C} \) by the formula

\[
X^* \left( x(s) \right) = X^{(2)a}(s) \frac{\partial}{\partial x^a} \left( x(s) \right).
\]

Thus \( X^* \left( x(s) \right) \) can be considered as the projection of the vector field \( X \left( x(s), v^{(1)}x'(s), y^{(2)a}(s) \right) \) on the tangent space \( TM \) at \( x(s) \in \mathcal{C} \). As an example, \( \frac{\partial}{\partial y^{(2)a}} \) is a projectable vector field. Also we shall see later that a Frenet frame for a curve on the manifold \( \tilde{Osc}^2 M \) contains only projectable vector fields.

Let \( D\Gamma (N) \), the canonical \( N \)-linear metric connection (5.2) and \( B\Gamma (N) \), the \( N \)-linear connection on Berwald type from (5.4), and \( \tilde{\nabla}_i, \tilde{\nabla}_j, (i = 0, 1, 2) \) are the covariant derivatives of these \( N \)-linear connections.

**Proposition 5.2.** The covariant derivatives of any projectable vector field \( X \) in the direction of \( \frac{\delta}{\delta v^{(1)}} \) or \( \frac{\partial}{\partial v^{(2)}} \) with respect to \( D\Gamma (N) \) and \( B\Gamma (N) \) vanish identically on \( \tilde{Osc}^2 \mathcal{C} \), that is we have

\[
\left( \nabla_{\frac{\delta}{\delta v^{(1)}}} X \right) \left( x(s), v^{(1)}x'(s), y^{(2)a}(s) \right) = 0,
\]

\[
\left( \nabla_{\frac{\partial}{\partial v^{(2)}}} X \right) \left( x(s), v^{(1)}x'(s), y^{(2)a}(s) \right) = 0, \forall s \in (-\varepsilon, \varepsilon),
\]

where \( \nabla_i \) are \( \nabla_{c(i)} \) or \( \nabla_{b(i)} \), \( (i = 0, 1, 2) \).

**Proof.** We have

\[
X^{(0)a}(s) \frac{\delta}{\delta x^a} + X^{(1)a}(s) \frac{\delta}{\delta y^{(1)a}} + X^{(2)a}(s) \frac{\partial}{\partial y^{(2)a}},
\]

\[
x(s), v^{(1)}x'(s), y^{(2)a}(s) \right).
\]

\[
X^* \left( x(s) \right) = X^{(2)a}(s) \frac{\partial}{\partial x^a} \left( x(s) \right).
\]
Gauss-Weingarten and Frenet equations in the theory of the homogeneous lift.

where

\[ X^{(i)a}(s) \big|_{i} = X^{(i)b}(s) x^{c} C_{(ij)}^{a} b_{c} = 0, \]

since for the above N-linear connections we have either

\[ C_{(ij)}^{a} b_{c} = g^{a} b_{c} \text{ or } C_{(ij)}^{a} b_{c} = 0, \]

where \( i \in \{0, 1, 2\} \), \( j \in \{1, 2\} \).

Hence, in particular, we have

\[ \nabla_{(i)} \frac{\partial}{\partial y^{(2)a}} = 0, \]

which enables us to state that the "vertical" covariant derivatives along \( C \) with respect to the canonical N-linear metric connection and of the Berwald type (5.4) do not provide any Frenet frame for \( C \). Hence we have to proceed with the horizontal covariant derivatives along \( C \).

From (2.5), we get

\[ \frac{\partial}{\partial s} = \frac{dx^{a}}{ds} \frac{\delta}{\delta x^{a}} + v^{(1)} \left( \frac{d^{2}x^{a}}{ds^{2}} + 2G^{a}(s) \right) \frac{\delta}{\delta y^{(1)a}} + v^{(2)} \left[ \frac{d^{2}x^{a}}{ds^{2}} + 2\tilde{G}^{a}(s, v^{(1)}) \right] \frac{\partial}{\partial y^{(2)a}}, \]

where \( \tilde{G}^{a}(s, v^{(1)}) = \frac{1}{2v^{(2)}} \left( \frac{ds}{d\delta} \frac{M^{a}}{2} + \frac{d^{2}x^{a}}{ds^{2}} G^{b}(s) v^{(1)} + \frac{d^{2}x^{a}}{ds^{2}} (v^{(1)})^{2} \right) \).

The canonical N-linear metric connection is the best choice for studying the geometry of curves in the manifold \( \tilde{Osc}^{2}M \). First, by direct calculation we get

\[ \nabla_{(i)} \frac{\partial}{\partial v^{(2)}} = \left( \frac{d^{2}x^{a}}{ds^{2}} + 2G^{a}(s) \right) \frac{\partial}{\partial y^{(2)a}}, \]

On the other hand, using (5.7) and taking into account that \( D\Gamma(N) \) is a metric N-linear connection we obtain

\[ \nabla_{(i)} \frac{\partial}{\partial v^{(2)}} = 0, (i = 0, 1, 2). \]

As a consequence of (5.15) we may set

\[ \nabla_{(i)} \frac{\partial}{\partial v^{(2)}} = k_{1} N_{1}, \]

where \( N_{1} \in V_{2}T(Osc^{2}C)^{\perp} \), and

\[ k_{1} = \left| \nabla_{(i)} \frac{\partial}{\partial v^{(2)}} \right|, \]
for \( i = 0, 1, 2 \) and \( \|X\| = \mathcal{G}(X, X), \forall X \in \mathcal{X}\left(\widehat{\text{Osc}}^2 M\right) \). By (5.14) we infer that

\[
k_1 = \left\{ g_{ab}(s)(x^n + 2G^n(s))(x^n + 2G^n(s)) \right\}^{1/2}
\]

(5.18)

and it call the geodesic curvature (first curvature) function of \( C \). If \( k_1(s) \neq 0, \forall s \in (-\epsilon, \epsilon) \) we call

\[
N_1 = \frac{1}{k_1(s)} (x^n + 2G^n(s)) \frac{\partial}{\partial y^{(2)a}} = N_1(s) \frac{\partial}{\partial y^{(2)a}}, \quad (5.19)
\]

the principal (first) normal of \( C \). Clearly, \( N_1 \) is a projectable vector field along \( C \). Actually, this is a consequence of the following general result.

**Proposition 5.3.** The covariant derivatives of a projectable vector field \( X \) along \( C \) with respect to the canonical \( N \)-linear metric connection in the direction of \( \frac{\partial}{\partial s} \) is a projectable vector field too, given by

\[
\nabla_{(i)} \frac{\partial}{\partial s} X = \left[ \frac{dX^{(0)a}}{ds} + X^{(0)b}S^a_b(s) \right] \frac{\delta}{\delta x^a} + \\
+ \left[ \frac{dX^{(1)a}}{ds} + X^{(1)b}S^a_b(s) \right] \frac{\delta}{\delta y^{(1)a}} + \\
+ \left[ \frac{dX^{(2)a}}{ds} + X^{(2)b}S^a_b(s) \right] \frac{\partial}{\partial y^{(2)a}}, \quad (i = 0, 1, 2)
\]

(5.20)

where

\[
S^a_b(s) = G^a_b(s) + (x^n + 2G^n(s)) g^{a,c}_b(s).
\]

(5.21)

**Proof.** The assertion follows by direct calculation using (5.13), (5.9) and (5.11). \( \Box \)

Next, suppose that \( n+1 > 2 \). Since \( \mathcal{D} \Gamma(N) \) is the canonical \( N \)-linear metric connection, from \( \mathcal{G}(N_1, N_1) = 1 \) and \( \mathcal{G}\left( \frac{\partial}{\partial y^{(1)}}, N_1 \right) = 0 \) using (5.16) we deduce that

\[
\mathcal{G}\left( \nabla_{(i)} \frac{\partial}{\partial s} N_1, N_1 \right) = 0, \quad \mathcal{G}\left( \nabla_{(i)} \frac{\partial}{\partial s} N_1, \frac{\partial}{\partial y^{(2)}} \right) = -k_1, \quad (i = 0, 1, 2).
\]

Hence we may set

\[
\nabla_{(i)} \frac{\partial}{\partial s} N_1 = -k_1 \frac{\partial}{\partial y^{(2)}} + N,
\]

(5.22)

where \( N \in V_2Osc^2C \).

Thus we may define the next function

\[
k_2 = \left\| k_1(s) \frac{\partial}{\partial y^{(2)}} + \nabla_{(i)} \frac{\partial}{\partial s} N_1 \right\|.
\]
Then by straightforward calculation using (5.19) and (5.20) we infer that

\[ k_2 = \left\{ g_{ab} \left( k_1 x^a + N_1^a + N_1^c S^a_c \right) \left( k_1 x^b + N_1^b + N_1^d S^a_d \right) \right\}^{1/2}. \]  

(5.23)

The function \( k_2 \) is called the **second curvature function** of \( C \).

If \( k_2(s) \neq 0, \forall (-\varepsilon, \varepsilon) \), we define the vector field

\[ N_2 = \frac{1}{k_2(s)} \left( k_1(s) \frac{\partial}{\partial v(2)} + \nabla_{(i)} \frac{\partial}{\partial s} N_1 \right). \]

Hence, (5.22) becomes

\[ \nabla_{(i)} \frac{\partial}{\partial s} N_1 = -k_1 \frac{\partial}{\partial v(2)} + k_2(s) N_2(s), (i = 0, 1, 2). \]

We suppose inductively that there exist orthonormal projectable vector fields \( \left\{ N_0 = \frac{\partial}{\partial v(2)}, N_1, ..., N_j \right\} \) and nowhere zero curvature functions \( \left\{ k_1, k_2, ..., k_j \right\}, 1 \leq j \leq n, \) such that the following equations hold

\[ (F_1) \ \nabla_{(i)} \frac{\partial}{\partial s} N_0 = k_1 N_1, \]

\[ (F_2) \ \nabla_{(i)} \frac{\partial}{\partial s} N_1 = -k_1 N_0 + k_2 N_2, \]  

\[ ... ... ... ... ... ... \]

\[ (F_j) \ \nabla_{(i)} \frac{\partial}{\partial s} N_{j-1} = -k_{j-1} N_{j-2} + k_j N_j, (i = 0, 1, 2). \]

Then by using the Proposition 5.3 and following a proof similar to that of the Finsler case (cf. Bejancu, Farran [3], p.156), for any \( j < n \) we obtain

\[ (F_{j+1}) \ \nabla_{(i)} \frac{\partial}{\partial s} N_j = -k_j N_{j-1} + k_{j+1} N_{j+1}, \]

where

\[ k_{j+1} = \left\{ g_{ab} \left( k_j N^a_{j-1} + N_j^a + N_j^c S^a_c \right) \left( k_j N^b_{j-1} + N_j^b + N_j^d S^a_d \right) \right\}^{1/2}. \]  

(5.25)

Moreover, the system of vector fields \( \left\{ N_0, N_1, ..., N_j \right\} \) is an orthonormal set of projectable vector fields along \( C \).

If \( j = n \), then \( \left\{ N_0, N_1, ..., N_j \right\} \) is an orthonormal basis of \( \mathcal{X}V_2T \left( \text{Osc}^2 C \right)^\perp \). As \( \nabla_{(i)} \frac{\partial}{\partial s} N_n \)

is orthogonal to \( N_n \), the equation \((F_{j+1})\) becomes

\[ (F_{n+1}) : \nabla_{(i)} \frac{\partial}{\partial s} N_n = -k_n N_{n-1}, (i = 0, 1, 2). \]
We call the system of vector fields \( \{N_0, N_1, ..., N_n\} \) and the equations \( \{(F_1), ..., (F_{n+1})\} \) the Frenet frame and the Frenet equations along \( C \) respectively. If \( k_j = 0 \) on \((-\varepsilon, \varepsilon)\), for some \( j < n \), then we can not define \( N_j \). Thus the equation \( (F_j) \) becomes

\[
(F_j)': \nabla_{\xi} N_{j-1} = -k_{j-1}N_{j-2}, (i = 0, 1, 2).
\]

Thus, in the case in which there exist nowhere zero curvature functions \( \{k_1, k_2, ..., k_{j-1}\} \) on \((-\varepsilon, \varepsilon)\) and \( k_j \) is everywhere zero on \((-\varepsilon, \varepsilon)\), then we have constructed the Frenet frame \( \{N_0, N_1, ..., N_{j-1}\} \) satisfying the Frenet equations \( \{(F_1), ..., (F_{j-1}), (F_j)\}' \). We obtain the following fundamental theorem for curves in the manifold \( \tilde{Osc}^2 M \).

**Theorem 5.4.** Let \( (x_0, y_0^{(1)}, y_0^{(2)}) = (x_0^a, y_0^{(1)a}, y_0^{(2)a}) \) be a fixed point of the manifold \( \tilde{Osc}^2 M \), \( \{V_0, V_1, ..., V_n\} \) an orthonormal basis of \( V_2T(\tilde{Osc}^2 M) \) and \( k_1, k_2, ..., k_n : (-\varepsilon, \varepsilon) \rightarrow R \) be everywhere positive smooth functions. Then exists a unique curve \( C \) on given by equations \( x^a = x^a(s), y^{(1)a} = y^{(1)a}(s), y^{(2)a} = y^{(2)a}(s), s \in (-\varepsilon, \varepsilon) \), where \( s \) is the arc length parameter of \( C \), such that \( (x^a(0), y^{(1)a}(0), y^{(2)a}(0)) = (x_0^a, y_0^{(1)a}, y_0^{(2)a}) \) and \( k_1, k_2, ..., k_n \) are the curvature functions of \( C \) with respect to the Frenet frame \( \{N_0, N_1, ..., N_n\} \) which satisfies \( \tilde{N}_h(0) = V_h \), \( h \in \{0, ..., n\} \).

**Proof.** We can use the Theorem 2.1,[3], p.158.

**Remark 5.5.**

1. If we consider the homogenous lift \( \mathcal{G} \), (2.9), we obtain

\[
0^\mathcal{G} \left( \frac{\partial}{\partial v^{(2)}}, \frac{\partial}{\partial v^{(2)}} \right) = p^4
\]

and (5.15) becomes

\[
\mathcal{G} \left( \nabla_\xi \frac{\partial}{\partial v^{(2)}}, \frac{\partial}{\partial v^{(2)}} \right) = 0.
\]

2. The curvature geodesic functions become

\[
0^{k_{j+1}} = \left\{ g_{ab}(s) \left( k_j N_{j-1}^a + N_{j}^{(a) \alpha} + N_{j}^{(b) \alpha} S_{\alpha}^c \right) \left( k_j N_{j-1}^b + N_{j}^{(b) \alpha} + N_{j}^{(d) \alpha} S_{\alpha}^d \right) \right\}^{1/2}
\]

\[
= p^2 k_{j+1}, (j = 0, 1, ..., n - 1).
\]

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Gauss-Weingarten and Frenet equations in the theory of the homogeneous lift.

References


