ON LIFTS OF LEFT-INVARINT HOLOMORPHIC VECTOR FIELDS IN COMPLEX LIE GROUPS

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Abstract

In this paper the complete, vertical and horizontal lifts of left invariant holomorphic vector fields to the holomorphic tangent bundle $T^{1,0}G$ of a complex Lie group $G$ are studied.

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1 Introduction

The study of complete, vertical and horizontal lifts of left-invariant vector fields on both tangent and tensor bundles of $(2,0)$ type over a real Lie group was initiated and intensively studied in [6, 7, 8]. The aim of this note is to obtain a complex analytic version of these notions on the holomorphic tangent bundle of a complex Lie group.

The paper is organized as follows. In the second section we present the complex Lie group structure of the holomorphic tangent bundle $T^{1,0}G$ of a complex Lie group $G$ and we construct the complete and vertical lifts of left-invariant holomorphic vector fields on $T^{1,0}G$. In the third section we consider a holomorphic horizontal distribution on $T^{1,0}G$ defined by a linear holomorphic connection on $G$. In the last section we construct horizontal lifts of left-invariant holomorphic vector fields on $T^{1,0}G$ and we give necessary and sufficient conditions for the horizontal lifts of left-invariant holomorphic vector fields to be left-invariant on $T^{1,0}G$.

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2 The complex Lie group structure of the holomorphic tangent bundle $T^{1,0}G$

Let $G$ be a complex Lie group. Let us denote by $(u,v) \to w = uv$ the composition law of the complex Lie group $G$, by $S$ the inverse mapping $u \to u^{-1}$ and by $R_a$ and $L_a$, the left and right transitions of the group, respectively, where $a \in G$. These mappings are holomorphic, see [5]. We can now define a composition law "$\circ$" on $T^{1,0}G$.

Let $U = (u, \eta_u), V = (v, \eta_v)$ be two holomorphic vector fields on $T^{1,0}G$. Then

$$(u, \eta_u) \circ (v, \eta_v) = (uv, L_s(u)\eta_v + R_s(v)\eta_u)$$

defines a holomorphic composition law on $T^{1,0}G$. In local coordinates, we have

$$w^k = \varphi^k(u^i, v^j), \quad \eta^i = L^i_s(u)\eta^i_v + R^i_s(v)\eta^i_u.$$  \hspace{1cm} (2)

**Theorem 1.** The holomorphic tangent bundle $T^{1,0}G$ is a complex Lie group with respect to the composition law defined in (1).

**Proof.** The identity of the group $T^{1,0}G$ is $E = (e, 0)$, where $e$ is the identity of $G$. Indeed, one has

$$(u, \eta_u) \circ (e, 0) = (ue, L_s(u) \cdot 0 + R_s(e) \cdot \eta_u) = (u, \eta_u).$$

Similarly, $(e, 0) \circ (u, \eta_u) = (u, \eta_u)$.

For the inverse of $U \in T^{1,0}G$, $U \circ V = E$ yields $uv = e$ and $L_s(u)\eta_v + R_s(v)\eta_u = 0$. These imply $v = u^{-1}$ and $\eta_v = -L_s^{-1}(u)R_s^{-1}(u)\eta_u = S_s\eta_u$. Therefore,

$$U^{-1} = (u^{-1}, S_s\eta_u),$$

where $S_s = -L_s^{-1}(u)R_s^{-1}(u)$.

In order to prove the associativity of the composition law (1), one has, on the one hand,

$$(u, \eta_u) \circ (v, \eta_v) = (uv, L_s(u)\eta_v + R_s(v)\eta_u) = (uv, \tau_{uv}),$$

and, on the other hand,

$$(v, \eta_v) \circ (z, \eta_z) = (vz, L_s(v)\eta_z + R_s(z)\eta_v) = (vz, \zeta_{uv}),$$

and, on the other hand,

$$(u, \eta_u) \circ ((v, \eta_v) \circ (z, \eta_z)) = (u(vz), L_s(u)\zeta_{uv} + R_s(vz)\eta_u).$$

But $G$ is a complex Lie group and by using $L_s(uv) = L_s(u)L_s(v)$, $R_s(uv) = R_s(u)R_s(v)$ and $L_s(u)R_s(v) = R_s(v)L_s(u)$, the associativity is also proved. Therefore, $T^{1,0}G$ is a complex Lie group with the composition law (1). \qed
Remark 1. Let \( \omega_u \in (T^{1,0}G)^* \) be a holomorphic 1-form on \( G \). One has
\[
\omega_u(\eta_u) = \omega_u(S_u \eta_u^{-1}) = S^* \omega_u(\eta_u^{-1}) = \omega_{u^{-1}}(\eta_u^{-1}),
\]
such that \( \omega_u(\eta_u) = \omega_{u^{-1}}(S_u \eta_u) \). Therefore,
\[
(u, \omega_u)^{-1} = (u^{-1}, S^* \omega_u).
\]
is the inverse of \( (u, \omega_u) \in (T^{1,0}G)^* \).

Let us now extend the notion of left-invariance on Lie groups to holomorphic vector fields on complex Lie groups. Recall that a holomorphic vector field \( \xi \) on \( G \) is called left-invariant if
\[
L^*(a) \xi(u) = \xi(au)
\]
for any \( u \in G \). For \( u = e \), we have
\[
\xi(a) = L^*_a(z),
\]
where \( z \) is a holomorphic vector field on the complex Lie group \( G \). In local coordinates,
\[
\xi^i(a) = L^i_j(a) z^j,
\]
where
\[
L^i_j(a) = (\partial_u \varphi^j(a, u))_e,
\]
and \( \partial_u = \partial_j = \frac{\partial}{\partial u^j} \).

Now we can apply these considerations to the complex Lie group \( T^{1,0}G \). If we denote by \( L(A) \) the matrix of the holomorphic composition law (1), locally given by (2), then a left-invariant holomorphic vector field \( \xi \) satisfies
\[
\xi(A) = L^*_a Z,
\]
where \( A \in T^{1,0}G \) and \( Z \) is a holomorphic vector field. If we put \( U = A \) and \( V = E \) in (1), its Jacobi matrix is

\[
L_s(A) = \begin{pmatrix}
L^*_s(a) & 0 \\
(\partial_u R^i_s(u))_e \eta_a & L^*_s(a)
\end{pmatrix}.
\]

(5)

From (2), we obtain the following local representations:
\[
L^*_s(A) = (L^i_j(a)), \quad (\partial_u R^i_s(u))_e \eta_a = (R^i_{sj}(a) \eta^j_a),
\]
where
\[
R^i_{sj}(a) = \left( \frac{\partial^2 \varphi^i(a, u)}{\partial u^s \partial a^j} \right)_{u=e}.
\]

(7)

As a consequence, one has
\[
\xi(A) = L^i_j(a) z^j \partial_i + [R^k_{si}(a) \eta^i_a z^s + L^k_i(a) \tilde{z}^i] \tilde{\partial}_k,
\]

(8)
where $\hat{\partial}_k = \frac{\partial}{\partial x^k}$ and $(z^i, \dot{z}^j)$ are the components of $Z \in T^{1,0}G$.

Let us denote by $E_\alpha(A) = (e_i(A), \dot{e}_j(A))$, where

$$e_i(A) = L^i_j(a)\partial_j + R^k_{ij}(a)\eta^i_k\hat{\partial}_k, \quad \dot{e}_j(A) = L^s_j(a)\dot{\hat{\partial}}_s$$

are called the complete and vertical lifts of $A$, respectively.

With these notations, formula (8) suggests the following decomposition:

$$Z(A) = z^i e_i(A) + \dot{z}^j \dot{e}_j(A).$$

A similar calculation as in the real case, see [6], leads to the following expression of Lie brackets of holomorphic vector fields given by (9):

$$[e_i, e_j] = c^k_{ij} e_k, \quad [e_i, \dot{e}_j] = c^k_{ij} \dot{e}_k, \quad [\dot{e}_i, \dot{e}_j] = 0,$$

where $c^k_{ij}$ are the usual structure constants of the complex Lie group $G$.

Also, the structure equations of the complex Lie group $T^{1,0}G$ with respect to the dual basis $\{\bar{\omega}^i = (\omega^i)^v, \bar{\omega}^{n+i} = (\omega^i)^c\}$ of $\{e_i, \dot{e}_j\}$, given by vertical and complete lifts of the 1-forms $\{\omega^i\}$ on $G$, can be expressed as follows:

$$\partial \bar{\omega}^i = -\frac{1}{2} c^i_{jk} \bar{\omega}^j \wedge \bar{\omega}^k, \quad \partial \bar{\omega}^{n+i} = -\frac{1}{2} c^i_{jk} \bar{\omega}^j \wedge \bar{\omega}^{n+k}.$$  

3 Holomorphic connections on $T^{1,0}G$

Let us consider the holomorphic projection $\pi : T^{1,0}G \to G$. Its holomorphic tangent map $\pi_* : T^{1,0}(T^{1,0}G) \to T^{1,0}G$ is a morphism of holomorphic tangent bundles, which maps a holomorphic vector $U$ at point $Z \in T^{1,0}G$ to a holomorphic vector $u = \pi_* U$ at point $\pi(Z)$. As a result, we have the vertical subbundle

$$V^{1,0}(T^{1,0}G) = \ker \pi_* \subset T^{1,0}(T^{1,0}G),$$

which is holomorphic, and its sections are called vertical vector fields on $T^{1,0}G$. Vertical subspaces make up an involutive distribution on the manifold $T^{1,0}G$.

The holomorphic tangent bundle $T^{1,0}G$ is said to be endowed with a complex nonlinear connection if there is a complex distribution $H^{1,0}(T^{1,0}G)$ which is complementary to the vertical distribution, that is

$$T^{1,0}(T^{1,0}G) = H^{1,0}(T^{1,0}G) \oplus V^{1,0}(T^{1,0}G).$$

A horizontal distribution $H^{1,0}(T^{1,0}G)$ on the holomorphic tangent bundle $T^{1,0}G$ can be locally specified by the projected vector fields

$$\partial_i^H = \partial_i - N^j_i(Z)\hat{\partial}_j,$$

which are $\pi$-connected with the vector fields $\partial_i$ of the natural frame field on the base manifold $T^{1,0}G$. 
Lifts of left-invariant holomorphic vector fields

Generally, we notice that the horizontal distribution $H^1,0(T^1,0G)$ is not a holomorphic one. If the functions $N^j_i(z)$ depend linearly and uniformly on the fiber coordinates $\eta^j$, that is,

$$N^j_i(z^k, \eta^k_z) = N^j_i(z^k)\eta^i_z,$$

the connection is said to be linear. Thus, the linear connection is specified by the functions $N^j_i(z)$, called the components of the linear connection. If, moreover, the linear connection is holomorphic, then the horizontal distribution defined by it is a holomorphic one.

Since any complex Lie group is a complex parallelizable manifold, see [10], there are canonical linear holomorphic connections on it. Let us consider the left connection $\hat{\nabla}$ with respect to which the left-invariant vector fields are absolutely parallel:

$$\hat{\nabla}_{\partial_i} L^k_j(z) = (L^r_j \hat{\Gamma}^k_{ir} + \partial_i L^k_j(z))\partial_k = 0.$$

Thus, the coefficients of the left holomorphic connection have the form

$$\hat{\Gamma}^k_{ij}(z) = -\tilde{L}^r_j(z)\partial_i L^k_r(z)\partial_i \tilde{L}^r_j(z),$$

where $(\tilde{L}^r_j(z))$ is the inverse of the matrix $(L^r_j(z))$.

4 Horizontal and vertical lifts

Let $U = U^i \partial_i \in V^{1,0}_E(T^1,0G)$ be an arbitrary vertical holomorphic vector field, acted upon by the differential of the left translation (5). Then

$$U(Z) = L^i_j(z)U = L^i_j(z)U^j \partial_i,$$

which shows that $U(Z) \in V^{1,0}_Z(T^1,0G)$. Thus, we have

**Proposition 1.** The vertical distribution $V^{1,0}(T^1,0G) \subset T^1,0G$ is left-invariant.

In the following, we consider a holomorphic horizontal distribution defined by a linear holomorphic connection. Let $E_i(Z) = e_i^H(Z)$ be the horizontal lift of a left-invariant holomorphic vector field $e_i(z)$ on $G$. The mapping of the horizontal lift, i.e. the linear isomorphism $H : T^1,0zG \to H^1,0(T^1,0G)$, commutes with the differential of the left translation:

$$E_i(Z) = e_i^H(Z) = (L^i_j(z)\partial_i)^H = L^i_j(z)\partial_i^H.$$

We shall now analyze the conditions under which $E_i$ are left-invariant vector fields. The condition of left-invariance of $E_i$ is

$$E_i(AZ) = L^i_j(A)E_i(Z),$$

where $A \in T^1,0G$. In local coordinates with respect to the natural field of frames $E_i$, the left-invariance condition has the form

$$E_i(Z) = L^k_i(z)(\partial_k - N^j_{kl}(z)\eta^l_z \partial_j).$$
Then
\[ L_s(A)E_i(Z) = L^k_j(a)L^j_i(z)\partial_k - (R^l_{si}(a)\eta^s\partial_l^s L^j_i(z) + L^l_j(a)L^k_i(z)N^s_{kr}\eta^r)\partial_l. \] (14)

On the other hand,
\[ E_i(AZ) = L^k_i(az)(\partial_k - N^l_{kl}(az)L^l_j(a)\eta^l\partial_l). \] (15)

Note that formula (1) implies that
\[ \eta^l_{az} = L^l_s(a)\eta^s_z + R^l_{ai}(a)\eta^l_{a}. \]

Therefore,
\[ E_i(AZ) = L^k_i(az)\partial_k - (L^l_j(az)N^j_{kl}(az)L^l_s(a)\eta^s_z + L^l_j(az)N^l_{kl}(az)R^l_{ai}(a)\eta^l_{a})\partial_l. \] (16)

By setting \( X = E = (e, 0) \) in (14) and (15), one obtains
\[ L_s(A)E_i(E) = L^k_i(a)\partial_k - R^l_{ai}(a)\eta^l_{a}\partial_l \]
and
\[ E_i(A) = L^k_i(a)\partial_k - L^l_j(a)N^l_{ki}(az)\eta^l_{a}\partial_l. \]

Combining the last two formula yields
\[ R^l_{ai}(a)\eta^l_{a} = L^k_i(a)N^l_{ki}(a)\eta^l_{a}, \]
which in turn implies
\[ N^j_{is}(a) = -L^k_i(a)R^j_{ki}(a). \] (17)

Thus, we have

**Theorem 2.** A necessary and sufficient condition for the horizontal lifts of left-invariant holomorphic vector fields to be left-invariant is that the coefficients of the linear holomorphic connection are given by (17).

**Corollary 1.** The field of holomorphic frames \( E_i \) is the left-invariant field of frames of the holomorphic horizontal distribution \( H^{1,0}(T^{1,0}G) \).

Let us now consider the vertical vector fields \( \dot{E}_h(Z) = L^1_h(z)\partial_l. \) According to Proposition 1, we have

**Corollary 2.** The field of holomorphic frames \( \dot{E}_h \) is the left-invariant field of frames of the holomorphic vertical distribution \( V^{1,0}(T^{1,0}G) \).

Thus, we have constructed the left-invariant and adapted field of holomorphic frames \( E_A = (E_k, \dot{E}_h) \), where
\[ \begin{cases} E_k(Z) = L^1_k(z)\partial_k^H, \\ \dot{E}_h(Z) = L^1_h(z)\partial_l. \end{cases} \] (18)

Finally, by similar calculations as in the real case for the tensor bundle of type \((2, 0)\) of a Lie group, we obtain
Proposition 2. The Lie brackets of the vector fields defined in (18) are:

\[
[E_k, E_h] = c^i_{kh} E_i, \quad [E_k, \dot{E}_h] = c^i_{kh} \dot{E}_i, \quad [\dot{E}_k, \dot{E}_h] = 0,
\]

(19)

where \( c^i_{jk} \) are the usual constants structure of \( G \) and \( \dot{c}^r_{kh} = (\partial_i L^r_h(z))_e + N^r_{ij}(e) \).

Remark 2. From the first identity in (19) it follows that the holomorphic horizontal distribution defined by a linear holomorphic connection with the coefficients given by (17) is integrable.

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References


