ON A CLASS OF $LP$-SASAKIAN MANIFOLDS

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Abstract

The object of the present paper is to study projective curvature tensor in $LP$-Sasakian manifolds. $LP$-Sasakian manifolds satisfying $P.R = 0$, $R.P = 0$ and $P.S = 0$ are also considered. $\phi$-Ricci symmetric $LP$-Sasakian manifolds have been studied. In all the cases the manifold becomes an Einstein manifold. Next we study 3-dimensional $LP$-Sasakian manifold satisfying $divP = 0$. Finally, we construct an example of a 3-dimensional $LP$-Sasakian manifold which verifies our result.

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1 Introduction

In 1989 Matsumoto [7] introduced the notion of Lorentzian para-Sasakian manifolds. Then Mihai and Rosca [10] defined the same notion independently and they obtained several results in this manifold. $LP$-Sasakian manifolds have also been studied by Matsumoto and Mihai [8], Matsumoto, Mihai and Rosca [9], Mihai, Shaikh and De [11], De and Shaikh ([2],[4]), Ozgur [13] and many others. After the conformal curvature tensor the projective curvature tensor is an important tensor from the differential geometric point of view. Let $M$ be an $n$—dimensional Riemannian manifold. If there exists a one-to-one correspondence between each coordinate neighborhood of $M$ and a domain in Euclidian space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then $M$ is said to be locally projectively flat. For $n \geq 3$, $M$ is locally projectively flat if and only if the well known projective curvature tensor $P$ vanishes. Here $P$ is defined by [12]

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{n - 1}\{S(Y, Z)X - S(X, Z)Y\}, \quad (1.1)$$

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for $X, Y, Z \in T(M)$, where $R$ is the curvature tensor and $S$ is the Ricci tensor. In fact, $M$ is projectively flat (that is, $P = 0$) if and only if the manifold is of constant curvature (pp. 84-85 of [16]). Thus, the projective curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.

A Riemannian or a semi-Riemannian manifold is said to be semi-symmetric ([14],[6]) if $R(X, Y) \cdot R = 0$, where $R$ is the Riemannian curvature tensor and $R(X, Y)$ is considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors $X, Y$. If a Riemannian manifold satisfies $R(X, Y) \cdot P = 0$, then the manifold is said to be projectively semi-symmetric manifold.

Motivated by the above works we study some properties of projective curvature tensor in $LP$-Sasakian manifolds.

An $LP$-Sasakian manifold is said to be locally $\phi$-symmetric if

$$\phi^2((\nabla_X R)(Y, Z)W) = 0,$$

(1.2)

for all vector fields $X, Y, Z, W$ orthogonal to $\xi$. This notion was introduced for Sasakian manifolds by Takahashi [15]. Later in [1], Blair, Koufogiorgos and Sharma studied locally $\phi$-symmetric contact metric manifolds. In (1.2), if $X, Y, Z$ and $W$ are not horizontal vectors then we call the manifold globally $\phi$-symmetric.

An $LP$-Sasakian manifold is said to be $\phi$-Ricci symmetric if the Ricci operator $Q$ satisfies

$$\phi^2(\nabla_X Q)(Y) = 0,$$

(1.3)

for all vector fields $X, Y \in T(M)$ and the Ricci operator $Q$ is defined by $S(X, Y) = g(QX, Y)$, where $S$ is the Ricci tensor. If $X, Y$ are orthogonal to $\xi$, then the manifold is said to be locally $\phi$-Ricci symmetric. From the definition it follows that $\phi$-symmetric implies $\phi$-Ricci symmetric, but the converse, is not, in general true. $\phi$-Ricci symmetric Sasakian manifolds have been studied by De and Sarkar [3].

Again an $LP$-Sasakian manifold is called Einstein if the Ricci tensor $S$ is of the form $S = \lambda g$, where $\lambda$ is a constant.

The paper is organized as follows:

In section 2, some preliminary results are recalled. After preliminaries, we study $LP$-Sasakian manifolds satisfying $P \cdot R = 0$ and $R \cdot P = 0$. Section 4 deals with $LP$-Sasakian manifolds satisfying $P \cdot S = 0$. In the next section, we prove that an $n$-dimensional $LP$-Sasakian manifold is $\phi$-Ricci symmetric if and only if it is an Einstein manifold. Next we study 3-dimensional $LP$-Sasakian manifold satisfying $div P = 0$ and prove that in that case the manifold is a space form. Finally, we construct some examples of $LP$-Sasakian manifold which verifies our result.
2 Preliminaries

Let $M^n$ be an $n$-dimensional differentiable manifold endowed with a $(1,1)$ tensor field $\phi$, a contravariant vector field $\xi$, a covariant vector field $\eta$ and a Lorentzian metric $g$ of type $(0,2)$ such that for each point $p \in M$, the tensor $g_p: T_pM \times T_pM \to \mathbb{R}$ is a non-degenerate inner product of signature $(-, +, +, \ldots, +)$, where $T_pM$ denotes the tangent space of $M$ at $p$ and $\mathbb{R}$ is the real number space which satisfies

$$\phi^2(X) = X + \eta(X)\xi, \eta(\xi) = -1,$$

$$g(X, \xi) = \eta(X), g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$$

for all vector fields $X, Y$. Then such a structure $(\phi, \xi, \eta, g)$ is termed as Lorentzian almost paracontact structure and the manifold $M^n$ with the structure $(\phi, \xi, \eta, g)$ is called Lorentzian almost paracontact manifold [7]. In the Lorentzian almost paracontact manifold $M^n$, the following relations hold [7]:

$$\phi \xi = 0, \eta(\phi X) = 0,$$

$$\Omega(X, Y) = \Omega(Y, X),$$

where $\Omega(X, Y) = g(X, \phi Y)$.

Let $\{e_i\}$ be an orthonormal basis such that $e_1 = \xi$. Then the Ricci tensor $S$ and the scalar curvature $r$ are defined by

$$S(X, Y) = \sum_{i=1}^{n} \epsilon_i g(R(e_i, X)Y, e_i)$$

and

$$r = \sum_{i=1}^{n} \epsilon_i S(e_i, e_i),$$

where we put $\epsilon_i = g(e_i, e_i)$, that is, $\epsilon_1 = -1, \epsilon_2 = \cdots = \epsilon_n = 1.$

A Lorentzian almost paracontact manifold $M^n$ equipped with the structure $(\phi, \xi, \eta, g)$ is called Lorentzian paracontact manifold if

$$\Omega(X, Y) = \frac{1}{2} \{(\nabla_X \eta)Y + (\nabla_Y \eta)X\}.$$

A Lorentzian almost paracontact manifold $M^n$ equipped with the structure $(\phi, \xi, \eta, g)$ is called an LP-Sasakian manifold [7] if

$$(\nabla_X \phi)Y = g(\phi X, \phi Y)\xi + \eta(Y)\phi^2 X.$$

In an LP-Sasakian manifold the 1-form $\eta$ is closed. Also in [7], it is proved that if an $n$-dimensional Lorentzian manifold $(M^n, g)$ admits a timelike unit vector field $\xi$ such that the 1-form $\eta$ associated to $\xi$ is closed and satisfies

$$(\nabla_X \nabla_Y \eta)Z = g(X, Y)\eta(Z) + g(X, Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z),$$
then $M^n$ admits an LP-Sasakian structure. Further, on such an LP-Sasakian manifold $M^n (\phi, \xi, \eta, g)$, the following relations hold [7]:

\[
\eta(R(X, Y)Z) = [g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \quad (2.5)
\]

\[
S(X, \xi) = (n - 1)\eta(X), \quad (2.6)
\]

\[
S(\phi X, \phi Y) = S(X, Y) + (n - 1)\eta(X)\eta(Y), \quad (2.7)
\]

\[
R(X, Y)\xi = [\eta(Y)X - \eta(X)Y], \quad (2.8)
\]

\[
R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \quad (2.9)
\]

\[
(\nabla_X \phi)(Y) = [g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X], \quad (2.10)
\]

for all vector fields $X, Y, Z$, where $R, S$ denote respectively the curvature tensor and the Ricci tensor of the manifold. Also since the vector field $\eta$ is closed in an LP-Sasakian manifold, we have ([8],[7])

\[
(\nabla_X \eta)Y = \Omega(X, Y), \quad (2.11)
\]

\[
\Omega(X, \xi) = 0, \quad (2.12)
\]

\[
\nabla_X \xi = \phi X, \quad (2.13)
\]

for any vector field $X$ and $Y$.

We now give some examples of $LP$-Sasakian manifolds both in odd and even dimensions.

**Example 1** [9] Let $\mathbb{R}^5$ be the 5-dimensional real number space with a coordinate system $(x, y, z, t, s)$. Denoting

\[
\eta = ds - ydx - tdz, \quad \xi = \frac{\partial}{\partial s}, \quad g = \eta \otimes \eta - (dx)^2 - (dy)^2 - (dz)^2 - (dt)^2
\]

and

\[
\phi(\frac{\partial}{\partial x}) = -\frac{\partial}{\partial x} - y\frac{\partial}{\partial s}, \quad \phi(\frac{\partial}{\partial y}) = -\frac{\partial}{\partial y},
\]

\[
\phi(\frac{\partial}{\partial z}) = -\frac{\partial}{\partial z} - t\frac{\partial}{\partial s}, \quad \phi(\frac{\partial}{\partial t}) = -\frac{\partial}{\partial t}, \quad \phi(\frac{\partial}{\partial s}) = 0,
\]

the structure $(\phi, \xi, \eta, g)$ becomes an $LP$-Sasakian structure on $\mathbb{R}^5$. The metric tensor $g$ can be expressed by the matrix
On a class of LP-Sasakian manifolds

\[
g = \begin{pmatrix}
1 + y^2 & 0 & ty & 0 & -y \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-y & 0 & -1 & 0 & 0 \\
\end{pmatrix}.
\]

Example 2: Let \( \mathbb{R}^4 \) be the 4-dimensional real number space with a coordinate system \((x, y, z, t)\). In \( \mathbb{R}^4 \) we define

\[
\eta = dt - ydz - dx, \quad \xi = \frac{\partial}{\partial t},
\]

\[
g = e^{2t}(dx)^2 + e^{2t}(dy)^2 + (e^{2t} + y^2)(dz)^2 + ydz \otimes dx + ydx \otimes dz - ydz \otimes dt - ydt \otimes dz - \eta \otimes \eta,
\]

and

\[
\phi\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial x} + \frac{\partial}{\partial t}, \quad \phi\left(\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial y},
\]

\[
\phi\left(\frac{\partial}{\partial z}\right) = \frac{\partial}{\partial z}, \quad \phi\left(\frac{\partial}{\partial t}\right) = 0.
\]

Then it can be seen that the structure \((\phi, \xi, \eta, g)\) becomes an LP-Sasakian structure on \( \mathbb{R}^4 \). The metric \( g \) can be expressed by

\[
g = \begin{pmatrix}
e^{2t} - 1 & 0 & 0 & 1 \\
0 & e^{2t} & 0 & 0 \\
0 & 0 & e^{2t} & 0 \\
1 & 0 & 0 & -1 \\
\end{pmatrix}.
\]

3 \textbf{LP-Sasakian manifolds satisfying } \(P.R = 0\)

In view of (1.1) the projective curvature tensor of an \(n\)-dimensional LP-Sasakian manifold is given by

\[
P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y].
\]

Now from the above equation with the help of (2.6) and (2.9) we get

\[
P(\xi,V)\xi = 0 = P(V,\xi)\xi.
\]

In this section first we study LP-Sasakian manifolds satisfying

\[
(P(X,Y).R)(U,V)W = 0.
\]

Substituting \(Y = \xi\) in (3.3) we have

\[
\]
Putting $U = W = \xi$ in (3.4), we get
\begin{align*}
(P(X,\xi).R)(\xi,V)\xi &= P(X,\xi)R(\xi,V)\xi - R(P(X,\xi)\xi, V)\xi \\
&\quad - R(\xi, P(X,\xi)V)\xi - R(\xi,V)P(X,\xi)\xi \tag{3.5}
\end{align*}

Now,
\begin{align*}
P(X,\xi)R(\xi,V)\xi &= P(X,\xi)(V - \eta(V)\xi) \\
&= P(X,\xi)V - \eta(V)P(X,\xi)\xi \\
&= P(X,\xi)V. \tag{3.6}
\end{align*}

\begin{align*}
R(P(X,\xi)\xi,V)\xi &= 0. \tag{3.7}
\end{align*}

\begin{align*}
R(\xi, P(X,\xi)\xi)V)\xi &= P(X,\xi)V - g(P(X,\xi)V,\xi)\xi \\
&= P(X,\xi)V - g(X,V)\xi + \frac{1}{n-1}S(X,V)\xi. \tag{3.8}
\end{align*}

\begin{align*}
R(\xi,V)P(X,\xi)\xi &= 0. \tag{3.9}
\end{align*}

Using (3.6), (3.7), (3.8) and (3.9) in (3.5) we have
\begin{align*}
P(X,\xi)V - P(X,\xi)V + g(X,V)\xi - \frac{1}{n-1}S(X,V)\xi = 0. \tag{3.10}
\end{align*}

Taking inner product of (3.10) by $\xi$ we obtain
\begin{align*}
S(X,V) = (n - 1)g(X,V). \tag{3.11}
\end{align*}

Therefore the manifold is an Einstein manifold. Thus we can state the following:

**Theorem 3.1.** An $LP$-Sasakian manifold satisfying $P.R = 0$ is an Einstein manifold.

Next we study $LP$-Sasakian manifolds satisfying
\begin{align*}
(R(X,Y).P)(U,V)W = 0 \tag{3.12}
\end{align*}

Now substituting $Y = \xi$ in (3.12) we have
\begin{align*}
&\quad - P(U, R(X,\xi)V)W - P(U,V)R(X,\xi)W. \tag{3.13}
\end{align*}

Putting $U = W = \xi$ in (3.13) we have
\begin{align*}
(R(X,\xi).P)(\xi,\xi)\xi &= R(X,\xi)P(\xi,V)\xi - P(R(X,\xi)\xi, V)\xi \\
&\quad - P(\xi, R(X,\xi)V)\xi - P(\xi,V)R(X,\xi)\xi. \tag{3.14}
\end{align*}
On a class of $LP$-Sasakian manifolds

From (3.2) we obtain

\[ R(X, \xi)P(\xi, V)\xi = 0 = P(\xi, R(X, \xi)V)\xi. \]  

(3.15)

Again

\[
P(R(X, \xi)\xi, V)\xi = P(\eta(X)\xi - X, V)\xi \\
= -P(X, V)\xi + \eta(X)P(\xi, V)\xi \\
= -P(X, V)\xi. \]

(3.16)

and

\[
P(\xi, V)R(X, \xi)\xi = P(\xi, V)(\eta(X)\xi - X) \\
= \eta(X)P(\xi, V)\xi - P(\xi, V)X \\
= -P(\xi, V)X. \]

(3.17)

Using (3.15), (3.16), (3.17) in (3.14) we have

\[ P(X, V)\xi + P(\xi, V)X = 0. \]  

(3.18)

Taking the inner product of (3.18) by $\xi$ we obtain

\[ S(X, V) = (n - 1)g(X, V). \]  

(3.19)

Therefore the manifold is an Einstein manifold. Thus we can state the following:

**Theorem 3.2.** An $LP$-Sasakian manifold satisfying $R.P = 0$ is an Einstein manifold.

**4 $LP$-Sasakian manifolds satisfying $P.S = 0$**

In this section we study $LP$-Sasakian manifold satisfying $P.S = 0$. Therefore

\[ (P(X, Y).S)(U, V) = 0. \]

(4.1)

This implies

\[ S(P(X, Y)U, V) + S(U, P(X, Y)V) = 0. \]

(4.2)

Putting $Y = U = \xi$ in (4.2) we obtain

\[ S(P(X, \xi)\xi, V) + S(\xi, P(X, \xi)V) = 0. \]

(4.3)

Using (3.2) in (4.3), we have

\[ S(\xi, P(X, \xi)V) = 0. \]

(4.4)

This implies

\[ (n - 1)g(R(X, \xi)V - \frac{1}{n - 1}[S(\xi, V)X - S(X, V)\xi], \xi) = 0. \]

(4.5)
It follows that
\[
g(R(X,\xi)V,\xi) - \frac{1}{n-1}[(n-1)\eta(V)\eta(X) - S(X,V)] = 0. \tag{4.6}
\]
Therefore
\[
S(X,V) = (n-1)g(X,V). \tag{4.7}
\]
Hence the manifold is an Einstein manifold.
Conversely, the manifold is an Einstein manifold, that is, \( S(X,V) = \lambda g(X,V) \).
\[
(P(X,Y).S)(U,V) = S(P(X,Y)U,V) + S(U,P(X,Y)V) = \lambda[g(P(X,Y)U,V) + g(U,P(X,Y)V)]. \tag{4.8}
\]
Since
\[
g(P(X,Y)U,V) = -g(P(X,Y)V,U). \tag{4.9}
\]
Using (4.9) in (4.8) we have
\[
(P(X,Y).S)(U,V) = 0. \tag{4.10}
\]
Thus we can state the following:

**Theorem 4.1.** An LP-Sasakian manifold satisfies \( P.S = 0 \) if and only if it is an Einstein manifold.

## 5 \( \phi \)-Ricci symmetric LP-Sasakian manifolds

**Proposition 5.1.** An \( n \)-dimensional \( \phi \)-Ricci symmetric LP-Sasakian manifold is an Einstein manifold.

**Proof.** Let us assume that the manifold is \( \phi \)-Ricci symmetric. Then we have
\[
\phi^2(\nabla_X Q)(Y) = 0.
\]
Using (2.1) in the above, we get
\[
(\nabla_X Q)(Y) + \eta((\nabla_X Q)(Y))\xi = 0. \tag{5.1}
\]
From (5.1), it follows that
\[
g((\nabla_X Q)(Y),Z) + \eta((\nabla_X Q)(Y))\eta(Z) = 0, \tag{5.2}
\]
which on simplifying gives
\[
g(\nabla_X Q(Y),Z) - S(\nabla_X Y,Z) + \eta((\nabla_X Q)(Y))\eta(Z) = 0. \tag{5.3}
\]
Replacing $Y$ by $\xi$ in (5.3), we get
\[
g(\nabla_X Q(\xi), Z) - S(\nabla_X \xi, Z) + \eta((\nabla_X Q)(\xi))\eta(Z) = 0. \tag{5.4}
\]
By using (2.6) and (2.13) in (5.4), we obtain
\[
(n - 1)g(\phi X, Z) - S(\phi X, Z) + \eta((\nabla_X Q)(\xi))\eta(Z) = 0. \tag{5.5}
\]
Replacing $Z$ by $\phi Z$ in (5.5), we have
\[
S(\phi X, \phi Z) = (n - 1)g(\phi X, \phi Z). \tag{5.6}
\]
In view of (2.2) and (2.7), (5.6) becomes
\[
S(X, Z) = (n - 1)g(X, Z),
\]
which implies that the manifold is an Einstein manifold.

Now, since a $\phi$-symmetric manifold is $\phi$-Ricci symmetric, we have

**Corollary 5.1** A $\phi$-symmetric LP-Sasakian manifold is an Einstein manifold.

**Proposition 5.2.** If an $n$-dimensional LP-Sasakian manifold is an Einstein manifold, then it is $\phi$-Ricci symmetric.

**Proof.** Let us suppose that the manifold is an Einstein manifold. Then
\[
S(X, Y) = \alpha g(X, Y),
\]
where $S(X, Y) = g(QX, Y)$ and $\alpha$ is a constant. Hence $QX = \alpha X$. So, we have
\[
\phi^2(\nabla_X Q)(Y) = 0.
\]
This completes the proof.

In view of Proposition 5.1 and Proposition 5.2, we have

**Theorem 5.1.** An $n$-dimensional LP-Sasakian manifold is $\phi$-Ricci symmetric if and only if it is an Einstein manifold.
6 3-dimensional LP-Sasakian manifolds

Let us consider a 3-dimensional LP-Sasakian manifold. It is known that the conformal curvature tensor vanishes identically in the 3-dimensional Riemannian manifold. Thus we find

\[ R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y - \frac{r}{2}[g(Y,Z)X - g(X,Z)Y], \quad (6.1) \]

where \( Q \) is the Ricci operator, that is, \( g(QX,Y) = S(X,Y) \) and \( r \) is the scalar curvature of the manifold.

Putting \( Z = \xi \) in (6.1) and using (2.8) we have

\[ \eta(Y)QX - \eta(X)QY = \left( \frac{r}{2} - 1 \right)[\eta(Y)X - \eta(X)Y]. \quad (6.2) \]

Putting \( Y = \xi \) in (6.2) and using (2.1) and (2.6), we get

\[ QX = \frac{1}{2}[(r - 2)X + (r - 6)\eta(X)\xi], \quad (6.3) \]

that is,

\[ S(X,Y) = \frac{1}{2}[(r - 2)g(X,Y) + (r - 6)\eta(X)\eta(Y)]. \quad (6.4) \]

An LP-Sasakian manifold is said to be a space form if the manifold is a space of constant curvature.

**Lemma 6.1** A 3-dimensional LP-Sasakian manifold is a space form if and only if the scalar curvature \( r = 6 \).

**Proof.** Using (6.3) in (6.1), we get

\[ R(X,Y)Z = \left( \frac{r - 4}{2} \right)[g(Y,Z)X - g(X,Z)Y] + \left( \frac{r - 6}{2} \right)[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \quad (6.5) \]

From (6.5), the Lemma is obvious. \( \square \)

Let \( M \) be a 3-dimensional LP-Sasakian manifold with conservative projective curvature tensor [5], that is, \( divP = 0 \). Then its Ricci tensor is given by

\[ (\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z). \quad (6.6) \]

From this we obtain \( r = \text{constant} \).

From (6.4) we have

\[ (\nabla_X S)(Y,Z) = \frac{1}{2}[dr(X)\{g(Y,Z) + \eta(Y)\eta(Z)\} + (r - 6)\{\Omega(Y,X)\eta(Z) + \Omega(Z,X)\eta(Y)\}]. \quad (6.7) \]
Using (6.7) we get from (6.6)

\[ dr(X)\left[ \frac{1}{2}g(Y, Z) + \eta(Y)\eta(Z) \right] - dr(Y)\left[ \frac{1}{2}g(X, Z) + \eta(X)\eta(Z) \right] \\
+ (r - 6)\{\Omega(Z, X)\eta(Y) - \Omega(Z, Y)\eta(X)\} = 0. \] (6.8)

Taking a frame field and contracting over \( Y \) and \( Z \), we get

\[ dr(X) = (r - 6)\psi \eta(X), \] (6.9)

where \( \psi = \sum_{i=1}^{3} \Omega(e_i, e_i) = \text{trace} \phi \).

If we assume that \( \psi = \text{trace} \phi \neq 0 \), that is, \( \xi \) is not harmonic, then \( r = 6 \). So in view of Lemma 6.1 we state the following:

**Theorem 6.1.** A 3-dimensional LP-Sasakian manifold satisfying \( \text{div} \phi = 0 \) is a space form, provided the characteristic vector field \( \xi \) is not harmonic.

### 7 Examples

**Example 7.1:** We consider the 3-dimensional manifold \( M = \{(x, y, z) \in \mathbb{R}^3_1\} \), where \( (x, y, z) \) are standard coordinates of \( \mathbb{R}^3_1 \).

The vector fields

\[ e_1 = e^z \frac{\partial}{\partial y}, \quad e_2 = e^z \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right), \quad e_3 = \frac{\partial}{\partial z} \]

are linearly independent at each point of \( M \).

Let \( g \) be the Lorentzian metric defined by

\[ g(e_1, e_1) = g(e_2, e_2) = 1, \quad g(e_3, e_3) = -1, \]

\[ g(e_1, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0. \]

Let \( \eta \) be the 1-form defined by \( \eta(Z) = g(Z, e_3) \) for any vector field \( Z \in \chi(M) \).

Let \( \phi \) be the \((1, 1)\) tensor field defined by

\[ \phi(e_1) = -e_1, \quad \phi(e_2) = -e_2, \quad \phi(e_3) = 0. \]

Then using the linearity of \( \phi \) and \( g \) we have

\[ \eta(e_3) = -1, \]

\[ \phi^2 Z = Z + \eta(Z)e_3, \]

\[ g(\phi Z, \phi W) = g(Z, W) + \eta(Z)\eta(W) \]

for any vector fields \( Z, W \in \chi(M) \).

Then for \( e_3 = \xi \), the structure \((\phi, \xi, \eta, g)\) defines a Lorentzian paracontact structure on \( M \).
Let \( \nabla \) be the Levi-Civita connection with respect to the Lorentzian metric \( g \) and \( R \) be the curvature tensor of \( g \). Then we have 
\[
[e_1, e_2] = 0, [e_1, e_3] = -e_1
\]
and 
\[
\]

Taking \( e_3 = \xi \) and using Koszul’s formula for the Lorentzian metric \( g \), we can easily calculate
\[
\nabla_{e_1} e_1 = -e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = -e_1,
\]
\[
\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = -e_3, \quad \nabla_{e_2} e_3 = -e_2,
\]
\[
\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.
\]

From the above it can be easily seen that \( M_3(\phi, \xi, \eta, g) \) is an \( LP \)-Sasakian manifold. With the help of the above results it can be easily verified that
\[
R(e_1, e_2)e_3 = 0, \quad R(e_2, e_3)e_3 = -e_2, \quad R(e_1, e_3)e_3 = -e_1,
\]
\[
R(e_1, e_2)e_2 = e_1, \quad R(e_2, e_3)e_2 = -e_3, \quad R(e_1, e_3)e_2 = 0,
\]
\[
R(e_1, e_2)e_1 = -e_2, \quad R(e_2, e_3)e_1 = 0, \quad R(e_1, e_3)e_1 = -e_3.
\]

From the above expressions of the curvature tensor we obtain
\[
S(e_1, e_1) = g(R(e_1, e_2)e_2, e_1) - g(R(e_1, e_3)e_3, e_1) = 2.
\]

Similarly we have
\[
S(e_2, e_2) = 2, \quad S(e_3, e_3) = -2
\]
and
\[
S(e_i, e_j) = 0(i \neq j).
\]

Therefore,
\[
r = S(e_1, e_1) + S(e_2, e_2) - S(e_3, e_3) = 6.
\]

Therefore Theorem 6.1. is verified.

**Example 7.2:** Let us consider the 5-dimensional manifold \( \tilde{M} = \{(x, y, z, u, v) \in \mathbb{R}^5 : (x, y, z, u, v) \neq (0, 0, 0, 0, 0)\} \), where \( (x, y, z, u, v) \) are the standard coordinates in \( \mathbb{R}^5 \). The vector fields
\[
e_1 = e^z \frac{\partial}{\partial x}, \quad e_2 = e^{z-a} \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}, \quad e_4 = e^z \frac{\partial}{\partial u}, \quad e_5 = e^{z-v} \frac{\partial}{\partial v}
\]
are linearly independent at each point of \( \tilde{M} \) where \( a \) is scalar. Let \( \tilde{g} \) be the metric defined by
\[ \tilde{g}(e_i, e_j) = \begin{cases} 
1, & \text{for } i = j \neq 3, \\
0, & \text{for } i \neq j, \\
-1, & \text{for } i = j = 3. 
\end{cases} \]

Here \( i \) and \( j \) runs from 1 to 5. Let \( \eta \) be the 1-form defined by \( \eta(Z) = \tilde{g}(Z, e_3) \), for any vector field \( Z \) tangent to \( \tilde{M} \). Let \( \varphi \) be the \((1, 1)\) tensor field defined by

\[ \varphi e_1 = -e_1, \, \varphi e_2 = -e_2, \, \varphi e_3 = 0, \, \varphi e_4 = -e_4, \, \varphi e_5 = -e_5. \]

Then using the linearity property of \( \varphi \) and \( \tilde{g} \) we have

\[ \eta(e_3) = -1, \, \varphi^2 Z = Z + \eta(Z)e_3 \]

for any vector field \( Z \) tangent to \( \tilde{M} \). Thus for \( e_3 = \xi, \tilde{M}(\varphi, \xi, \eta, \tilde{g}) \) defines an almost para-contact metric manifold. Let \( \tilde{\nabla} \) be the Levi-Civita connection on \( \tilde{M} \) with respect to the metric \( \tilde{g} \). Then we have

\[
[e_1, e_2] = -ae^z e_2, \quad [e_1, e_3] = -e_1, \quad [e_1, e_4] = 0, \quad [e_1, e_5] = 0,
[e_2, e_3] = -e_2, \quad [e_2, e_4] = 0, \quad [e_2, e_5] = e_5, \quad [e_3, e_4] = e_4,
[e_3, e_5] = e_5, \quad [e_4, e_5] = -e^z e_5.
\]

Taking \( e_3 = \xi \) and using Koszul’s formula for \( \tilde{g} \), it can be easily calculated that

\[
\tilde{\nabla}_{e_1} e_1 = e_3, \quad \tilde{\nabla}_{e_1} e_2 = 0, \quad \tilde{\nabla}_{e_1} e_3 = -e_1, \quad \tilde{\nabla}_{e_1} e_4 = 0, \quad \tilde{\nabla}_{e_1} e_5 = 0,
\tilde{\nabla}_{e_2} e_1 = ae^z e_2, \quad \tilde{\nabla}_{e_2} e_2 = -ae^z e_3, \quad \tilde{\nabla}_{e_2} e_3 = -e_2, \quad \tilde{\nabla}_{e_2} e_4 = 0, \quad \tilde{\nabla}_{e_2} e_5 = 0,
\tilde{\nabla}_{e_3} e_1 = 0, \quad \tilde{\nabla}_{e_3} e_2 = 0, \quad \tilde{\nabla}_{e_3} e_3 = 0, \quad \tilde{\nabla}_{e_3} e_4 = 0, \quad \tilde{\nabla}_{e_3} e_5 = 0,
\tilde{\nabla}_{e_4} e_1 = 0, \quad \tilde{\nabla}_{e_4} e_2 = 0, \quad \tilde{\nabla}_{e_4} e_3 = -e_4, \quad \tilde{\nabla}_{e_4} e_4 = 0, \quad \tilde{\nabla}_{e_4} e_5 = 0,
\tilde{\nabla}_{e_5} e_1 = 0, \quad \tilde{\nabla}_{e_5} e_2 = 0, \quad \tilde{\nabla}_{e_5} e_3 = -e_5, \quad \tilde{\nabla}_{e_5} e_4 = e^z e_5, \quad \tilde{\nabla}_{e_5} e_5 = e_3 - e^z e_5.
\]

From the above calculations, we see the manifold under consideration satisfies \( \eta(\xi) = -1 \) and \( \tilde{\nabla}_X \xi = \varphi X \). Hence, \( \tilde{M} \) is an \( LP \)-Sasakian manifold.

**References**


