THE GAUSS-WEINGARTEN FORMULAE FOR THE HOMOGENEOUS LIFT TO THE OSCULATOR BUNDLE OF A FINSLER METRIC

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Abstract

In this article we present a study of the subspaces of the manifold OscM, the total space of the osculator bundle of a real manifold M. We obtain the induced connections of the canonical metrical N-linear connection determined by the homogeneous prolongation of a Finsler metric to the manifold OscM. We present the Gauss-Weingarten equations of the associated osculator submanifold.


Key words: nonlinear connection, linear connection, induced linear connection.

1 Introduction

The Sasaki $N$-prolongation $G$ to the osculator bundle without the null section $\tilde{\text{Osc}}M = \text{Osc}M \setminus \{0\}$ of a Finslerian metric $g_{ab}$ on the manifold $M$ given by

$$G = g_{ab}(x, y) \, dx^a \otimes dx^b + g_{ab}(x, y) \, \delta y^a \otimes \delta y^b$$  (*

is a Riemannian structure on $\tilde{\text{Osc}}M$, which depends only on the metric $g_{ab}$.

The tensor $G$ is not invariant with respect to the homothetis on the fibres of $\tilde{\text{Osc}}M$, because $G$ is not homogeneous with respect to the variable $y^a$.

In this paper, we use a new kind of prolongation $\tilde{G}$ to $\tilde{\text{Osc}}M$, ([8]), which depends only on the metric $g_{ab}$. Thus, $\tilde{G}$ determines on the manifold $\tilde{\text{Osc}}M$ a Riemannian structure which is 0-homogeneous on the fibres of $\text{Osc}M$.

Some geometrical properties of $\tilde{G}$ are studied: the canonical metrical $N$-linear connection, the induced linear connections etc.
2 Preliminaries

As far as we know the general theory of submanifolds (in particular the Finsler submanifolds or the complex Finsler submanifolds) is far from being settled ([1],[10], [3],[11], [12]). In [9] and [10] R. Miron and M. Anastasiei give the theory of subspaces in generalized Lagrange spaces. Also, in [6] and [5] R. Miron presented the theory of subspaces in higher order Finsler and Lagrange spaces respectively.

If $\tilde{M}$ is an immersed manifold in manifold $\tilde{M}$, a nonlinear connection on $\text{Osc}\tilde{M}$ induces a nonlinear connection $\tilde{N}$ on $\text{Osc}\tilde{M}$.

The d-tensor $G$ from (*) is not homogeneous with respect to the variable $y^a$. This is an inconvenience from the point of view of mechanics. Moreover, the physical dimensions of the terms of $G$ are not the same. This disadvantage was corrected by R. Miron. He took a new kind of prolongation $\tilde{G}$ to $\text{Osc}\tilde{M}$ of the fundamental tensor of a Finsler space, ([8]) (5), which depends only on the metric $g_{ab}$. Thus, $\tilde{G}$ determines on the manifold $\text{Osc}\tilde{M}$ a Riemannian structure which is $0$-homogeneous on the fibres of $\text{Osc}\tilde{M}$ and $p$ is a positive constant required by applications in order that the physical dimensions of the terms of $\tilde{G}$ be the same. He proved that there exist metrical N-linear connections with respect to the metric tensor $\tilde{G}$.

We take this canonical N-linear metric connection $D$ on the manifold $\text{Osc}\tilde{M}$ and obtain the induced tangent and normal connections and the relative covariant derivation in the algebra of d-tensor fields ([13],[16]).

In this paper we get the Gauss-Weingarten formulae of submanifold $\text{Osc}\tilde{M}$.

Let us consider $F^n = (M, F)$ a Finsler space ([10]), and $F : TM = \text{Osc}M \rightarrow \mathbb{R}$ the fundamental function. $F$ is a $C^\infty$ function on the manifold $\text{Osc}M$ and it is continuous on the null section of the projection $\pi : \text{Osc}M \rightarrow M$. The fundamental tensor on $F^n$ is

$$g_{ab}(x, y) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^a \partial y^b}, \quad \forall (x, y) \in \text{Osc}M.$$  

The lagrangian $F^2 (x, y)$ determines the canonical spray $S = y^a \frac{\partial}{\partial x^a} - 2G^a \frac{\partial}{\partial y^a}$ with the coefficients $G^a = \frac{1}{2} \gamma^a_{bc} (x, y) y^b y^c$, where $\gamma^a_{bc} (x, y)$ are the Christoffels symbols of the metric tensor $g_{ab} (x, y)$. The Cartan nonlinear connection $N$ of the space $F^n$ has the coefficients

$$N^a_b = \frac{\partial G^a}{\partial y^b}. \quad (1)$$

$N$ determines a distribution on the manifold $\text{Osc}M$, ([10],[9]), which is supplementary to the vertical distribution $V$. We have the next decomposition

$$T_w \text{Osc}M = N_w \oplus V_w, \forall w = (x, y) \in \text{Osc}M. \quad (2)$$
The adapted basis of this decomposition is \( \left\{ \frac{\delta}{\delta x^a}, \frac{\partial}{\partial y^a} \right\} \), \( (a = 1, \ldots, n) \) and its dual basis is \( (dx^a, \delta y^a) \), where

\[
\begin{cases}
\frac{\delta}{\delta x^a} = \frac{\partial}{\partial x^a} - N^b_a \delta \frac{\delta}{\delta y^b}, \\
\frac{\partial}{\partial y^a} = \frac{\partial}{\partial y^a}
\end{cases}
\]  

(3)

and

\[
\begin{cases}
dx^a = dx^a, \\
\delta y^a = dy^a + N^a_b dx^b.
\end{cases}
\]  

(4)

We use the next notations:

\[
\delta_a = \frac{\delta}{\delta x^a}, \quad \dot{\partial}_a = \frac{\partial}{\partial y^a}.
\]

The fundamental tensor \( g_{ab} \) determines on the manifold \( \tilde{\text{Osc}}M \) the homogeneous N-lift \( \mathcal{G},[8] \),

\[
\mathcal{G} = g_{ab}(x, y) dx^a \otimes dx^b + h_{ab}(x, y) \delta y^a \otimes \delta y^b,
\]

(5)

where

\[
h_{ab}(x, y) = \frac{p^2}{\|y\|^2} g_{ab}(x, y),
\]

(6)

\[\|y\|^2 = g_{ab}(x, y) y^a y^b.\]

This is homogeneous with respect to \( y \), and \( p \) is a positive constant required by applications in order that the physical dimensions of the terms of \( \mathcal{G} \) be the same.

Let \( \tilde{M} \) be a real, \( m \)-dimensional manifold, immersed in \( M \) through the immersion \( i : \tilde{M} \to M \). Locally, \( i \) can be given in the form

\[x^a = x^a(u^1, \ldots, u^m), \quad \text{rank} \left\| \frac{\partial x^a}{\partial u^\alpha} \right\| = m.\]

The indices \( a, b, c, \ldots \) run over the set \( \{1, \ldots, n\} \) and \( \alpha, \beta, \gamma, \ldots \) run on the set \( \{1, \ldots, m\} \). We assume \( 1 < m < n \). We take the immersed submanifold \( \text{Osc}M \) of the manifold \( \text{Osc}M \), by the immersion \( \text{Osci} : \text{Osc}M \to \text{Osc}M \). The parametric equations of the submanifold \( \text{Osc}M \) are

\[
\begin{cases}
x^a = x^a(u^1, \ldots, u^m), \quad \text{rang} \left\| \frac{\partial x^a}{\partial u^\alpha} \right\| = m \\
y^a = \frac{\partial x^a}{\partial u^\alpha} v^\alpha.
\end{cases}
\]  

(7)
The restriction of the fundamental function $F$ to the submanifold $\tilde{\text{Osc}}M$ is

$$\tilde{F}(u, v) = F(x(u), y(u, v))$$

and we call $\tilde{F}^m = (\tilde{M}, \tilde{F})$ the induced Finsler subspaces of $F^m$ and $\tilde{F}$ the induced fundamental function.

Let $B^a_{\alpha}(u) = \frac{\partial x^a}{\partial u^\alpha}$ and $g_{\alpha\beta}$ the induced fundamental tensor,

$$g_{\alpha\beta}(u, v) = g_{ab}(x(u), y(u, v)) B^a_{\alpha} B^b_{\beta}. \quad (8)$$

We obtain a system of d-vectors $\{B^a_{\alpha}, B^a_{\bar{\alpha}}\}$ which determines a moving frame $\mathcal{R} = \{(u, v); B^a_{\alpha}(u), B^a_{\bar{\alpha}}(u, v)\}$ in $\text{Osc}M$ along to the submanifold $\text{Osc}M$.

Its dual frame will be denoted by $\mathcal{R}^* = \{B^a_{\alpha}(u, v), B^a_{\bar{\alpha}}(u, v)\}$. This is also defined on an open set $\tilde{\pi}^{-1}(U) \subset \text{Osc}M$, $U$ being a domain of a local chart on the submanifold $\tilde{M}$.

The conditions of duality are given by:

$$B^a_{\beta} B^\beta_b = \delta^a_b, \quad B^a_{\beta} B^\beta_{\alpha} = 0, \quad B^a_{\alpha} B^\alpha_b = 0, \quad B^a_{\bar{\alpha}} B^\bar{\alpha}_b = \delta^a_b \quad (9)$$

$$B^a_{\alpha} B^\alpha_b + B^a_{\bar{\alpha}} B^\bar{\alpha}_b = \delta^a_b.$$

The restriction of the nonlinear connection $N$ to $\tilde{\text{Osc}}M$ uniquely determines an induced nonlinear connection $\tilde{N}$ on $\tilde{\text{Osc}}M$

$$\tilde{N}^a_{\alpha \beta} = B^a_{\alpha}(B^a_{0\beta} + N^a_{b} B^b_{\beta}). \quad (9)$$

The cobasis $(dx^i, \delta y^\alpha)$ restricted to $\text{Osc}M$ is uniquely represented in the moving frame $\mathcal{R}$ in the following form:

$$\begin{cases} dx^a = B^a_{\beta} du^\beta \\ \delta y^\alpha = B^a_{\alpha} \delta v^a + B^a_{\bar{\alpha}} K_{\beta}^\alpha du^\beta \quad (10) \end{cases}$$

where

$$K_{\beta}^\alpha = B^a_{\alpha} \left( B^a_{0\beta} + M^a_{b} B^b_{\beta} \right), \quad B^a_{0\beta} = B^a_{\alpha\beta} v^\alpha.$$

A linear connection $D$ on the manifold $\text{Osc}M$ is called metrical $N$-linear connection with respect to $\mathcal{G}$, if $DG = 0$ and $D$ preserves by parallelism the distributions $N$ and $V$. The coefficients of the N-linear connections $D\Gamma(N)$ will be denoted with $\left( \begin{array}{cccc} H^a_{(00)} & L^a_{(01)} & H^a_{(10)} & C^a_{(01)} \\ L^a_{(10)} & C^a_{(01)} & V^a_{(11)} & L^a_{(10)} \end{array} \right)$.

**Theorem 1.1** ([8]) There exist metrical $N$-linear connections $D\Gamma(N)$ on $\tilde{\text{Osc}}M$, with respect to the homogeneous prolongation $\mathcal{G}$, which depend only on the metric...
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\[ g_{ab}(x, y) \]. One of these connections has the "horizontal" coefficients

\[
\begin{align*}
H_{(00)}^a b c &= \frac{1}{2} g^{ad} (\delta_b g_{dc} + \delta_c g_{bd} - \delta_d g_{bc}) \\
V_{(10)}^a b c &= \frac{1}{2} h^{ad} (\delta_b h_{dc} + \delta_c h_{bd} - \delta_d h_{bc})
\end{align*}
\]

(11)

and the "vertical" coefficients:

\[
\begin{align*}
H_{(01)}^a b c &= \frac{1}{2} g^{ad} (\dot{\delta}_b g_{dc} + \dot{\delta}_c g_{bd} - \dot{\delta}_d g_{bc}) \\
V_{(11)}^a b c &= \frac{1}{2} h^{ad} (\dot{\delta}_b h_{dc} + \dot{\delta}_c h_{bd} - \dot{\delta}_d h_{bc})
\end{align*}
\]

(12)

It is called the Cartan metrical N-linear connection. This linear connection will be used throughout this paper.

For this N-linear connection, we have the operators \( H_D \) and \( V_D \) which are given by the following relations

\[
\begin{align*}
H D^a X &= dX^a + H^a b c X^b \\
V D^a X &= dX^a + V^a b c X^b
\end{align*}
\]

∀ \( X \in \mathcal{F}(\tilde{Osc}M) \) (13)

We call these operators the horizontal and vertical covariant differentials. The 1-forms which define these operators will be called the horizontal and vertical 1-form, where

\[
\begin{align*}
H^a b &= L_{(00)}^a b c dx^c + C_{(01)}^a b c dy^c \\
V^a b &= L_{(10)}^a b c dx^c + C_{(11)}^a b c dy^c
\end{align*}
\]

(14)

We have

**Theorem 1.2** [16] The d-tensors of torsion of the Cartan metrical N-linear connection \( D \) have the next expressions:

\[
\begin{align*}
\frac{H^a}{(00)b c} &= \frac{H^a}{(00)b c} - \frac{H^a}{(00)c b}, \\
\frac{V^a}{(10)b c} &= \frac{V^a}{(10)b c} - \frac{V^a}{(10)c b}
\end{align*}
\]

(15)
Theorem 1.3[16] The Cartan metrical N-linear connection $D$ has, in the adapted bases $\{\delta_a, \dot{\delta}_a\}$, the following $d$-tensors of curvature "horizontals"

\[
\begin{align*}
H^R_{(00)}{}^a{}_{bcd} &= \delta_d H_{(00)}{}^a{}_{bc} - \delta_c H_{(00)}{}^a{}_{bd} + H^f_{(00)}{}^a{}_{fc} L_{(00)}{}^a{}_{bd} - H^f_{(00)}{}^a{}_{fd} L_{(00)}{}^a{}_{bc} + \\
&\quad + H^f_{(01)}{}^a{}_{jl} R^f_{jd}, \\
H^P_{(10)}{}^a{}_{bcd} &= \dot{\delta}_d H_{(00)}{}^a{}_{bc} - \dot{\delta}_c H_{(00)}{}^a{}_{bd} + H^f_{(01)}{}^a{}_{jl} H_{(10)}{}^a{}_{fc} - H^f_{(01)}{}^a{}_{jd} H_{(10)}{}^a{}_{bc}, \\
H^S_{(10)}{}^a{}_{bcd} &= \dot{\delta}_d C_{(01)}{}^a{}_{bc} - \dot{\delta}_c C_{(01)}{}^a{}_{bd} + H^f_{(01)}{}^a{}_{jl} C_{(10)}{}^a{}_{fc} - H^f_{(01)}{}^a{}_{jd} C_{(10)}{}^a{}_{bc},
\end{align*}
\]

and the "verticals"

\[
\begin{align*}
V^R_{(01)}{}^a{}_{bcd} &= \delta_d V_{(10)}{}^a{}_{bc} - \delta_c V_{(10)}{}^a{}_{bd} + V^f_{(10)}{}^a{}_{fc} L_{(10)}{}^a{}_{bd} - V^f_{(10)}{}^a{}_{fd} L_{(10)}{}^a{}_{bc} + \\
&\quad + V^f_{(11)}{}^a{}_{jl} R^f_{jd}, \\
V^P_{(11)}{}^a{}_{bcd} &= \dot{\delta}_d V_{(10)}{}^a{}_{bc} - \dot{\delta}_c V_{(10)}{}^a{}_{bd} + V^f_{(11)}{}^a{}_{jl} V_{(10)}{}^a{}_{fc} - V^f_{(11)}{}^a{}_{jd} V_{(10)}{}^a{}_{bc}, \\
V^S_{(11)}{}^a{}_{bcd} &= \dot{\delta}_d C_{(11)}{}^a{}_{bc} - \dot{\delta}_c C_{(11)}{}^a{}_{bd} + V^f_{(11)}{}^a{}_{jl} C_{(11)}{}^a{}_{fc} - V^f_{(11)}{}^a{}_{jd} C_{(11)}{}^a{}_{bc}.
\end{align*}
\]

3 The relative covariant derivatives

Let $D\Gamma(N)$, the Cartan metrical N-linear connection of the manifold $OscM$. A classical method to determine the laws of derivation on a Finsler submanifold is the type of the coupling.

Theorem 2.1 The coupling of the N-linear connection $D$ to the induced nonlinear connection $N$ along $OscM$ is locally given by the set of coefficients $D\Gamma(N) =$
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\[
\begin{pmatrix}
H^a_{(00)b\delta}, & V^a_{(00)b\delta}, & H^a_{(01)b\delta}, & V^a_{(01)b\delta} \\
(10)b\delta, & (10)b\delta, & (11)b\delta, & (11)b\delta
\end{pmatrix},
\]

where

\[
\begin{align*}
L^a_{(00)b\delta} &= L^a_{(00)b\delta} B_d^d + C^a_{(01)b\delta} B_d^d K_d^\delta, \\
L^a_{(10)b\delta} &= L^a_{(10)b\delta} B_d^d + V^a_{(11)b\delta} B_d^d K_d^\delta, \\
C^a_{(01)b\delta} &= C^a_{(01)b\delta} B_d^d, \\
C^a_{(11)b\delta} &= C^a_{(11)b\delta} B_d^d.
\end{align*}
\]

(18)

**Definition 2.2** We call the **induced tangent connection** on \( \widetilde{OscM} \) by the metrical \( N \)-linear connection \( D \), the couple of operators \( D^H, D^V \) which are defined by

\[
\begin{align*}
D^H X^\alpha &= B^\alpha_b \tilde{D} X^b, \\
&\quad \text{for } X^\alpha = B^\alpha_\gamma X^\gamma \\
D^V X^\alpha &= B^\alpha_b \tilde{D} X^b,
\end{align*}
\]

where

\[
\begin{align*}
D^H X^\alpha &= dX^\alpha + X^\beta H^\alpha_\beta \\
D^V X^\alpha &= dX^\alpha + X^\beta V^\alpha_\beta
\end{align*}
\]

and \( H^\alpha_\beta, V^\alpha_\beta \) are called the **tangent connection 1-forms**.

We have

**Theorem 2.3** The tangent connections 1-forms are as follows:

\[
\begin{align*}
H^\alpha_\beta &= L^\alpha_{(00)\beta\delta} d\delta + C^\alpha_{(01)\beta\delta} d\delta, \\
V^\alpha_\beta &= L^\alpha_{(10)\beta\delta} d\delta + C^\alpha_{(11)\beta\delta} d\delta.
\end{align*}
\]

(19)
where

\[
\begin{align*}
\frac{H}{L}^{\alpha}_{\beta\delta} &= B^\alpha_d \left( B^d_{\beta\delta} + B^f_{\beta} \frac{H}{L}^{d}_{\delta} \right), \\
\frac{V}{L}^{\alpha}_{(10)\beta\delta} &= B^\alpha_d \left( B^d_{\beta\delta} + B^f_{\beta} \frac{V}{L}^{d}_{\delta} \right), \\
\frac{H}{C}^{\alpha}_{(01)\beta\delta} &= B^\alpha_d B^f_{\beta} \frac{H}{C}^{d}_{\delta}, \\
\frac{V}{C}^{\alpha}_{(11)\beta\delta} &= B^\alpha_d B^f_{\beta} \frac{V}{C}^{d}_{\delta},
\end{align*}
\]

(20)

Definition 2.4 We call the induced normal connection on \( \widetilde{\text{Osc}} M \) by the metrical \( N \)-linear connection \( D \), the couple of operators \( H^H, V^V \) which are defined by

\[
\begin{align*}
D^H X^\alpha &= B^\alpha_b \frac{D}{H} X^b, \\
D^V X^\alpha &= B^\alpha_b \frac{D}{V} X^b,
\end{align*}
\]

for \( X^a = B^a_\gamma X^\gamma \)

where

\[
\begin{align*}
D^H X^\alpha &= dX^\alpha + X^\beta \frac{H}{\omega}^\alpha_{\beta}, \\
D^V X^\alpha &= dX^\alpha + X^\beta \frac{V}{\omega}^\alpha_{\beta}
\end{align*}
\]

and \( \frac{H}{\omega}, \frac{V}{\omega} \) are called the normal connection 1-forms.

We have

Theorem 2.5 The normal connections 1-forms are as follows:

\[
\begin{align*}
\frac{H}{\omega}^\alpha_{\beta} &= \frac{H}{L}^{\alpha\beta}_{(00)\delta} du^\delta + \frac{H}{C}^{\alpha\beta}_{(01)\delta} dv^\delta, \\
\frac{V}{\omega}^\alpha_{\beta} &= \frac{V}{L}^{\alpha\beta}_{(10)\delta} du^\delta + \frac{V}{C}^{\alpha\beta}_{(11)\delta} dv^\delta,
\end{align*}
\]

(21)

where
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\[ H_L^{(00)\alpha\beta\gamma} = B_d^\alpha \left( \frac{\delta B^d_{\beta}}{\delta u^\delta} + B^d_{\beta(00)\delta} \right), \]

\[ V_L^{(00)\alpha\beta\gamma} = B_d^\alpha \left( \frac{\delta B^d_{\beta}}{\delta u^\delta} + B^d_{\beta(00)\delta} \right), \]

\[ H_C^{(01)\alpha\beta\gamma} = B_d^\alpha \left( \frac{\partial B^d_{\beta}}{\partial v^\delta} + B^d_{\beta(01)\delta} \right), \]

\[ V_C^{(01)\alpha\beta\gamma} = B_d^\alpha \left( \frac{\partial B^d_{\beta}}{\partial v^\delta} + B^d_{\beta(01)\delta} \right). \]

Now, we can define the relative (or mixed) covariant derivatives \( H \nabla \) and \( V \nabla \).

**Theorem 2.6** The relative covariant (mixed) derivatives in the algebra of mixed d-tensor fields are the operators \( H \nabla \), \( V \nabla \) for which the following properties hold:

- \( H \nabla f = df \), \( \forall f \in \mathcal{F}(\tilde{\text{Osc}}\tilde{M}) \)
- \( V \nabla f = df \), \( \forall f \in \mathcal{F}(\tilde{\text{Osc}}\tilde{M}) \)

\[ H \nabla X^a = \tilde{D}X^a, \quad V \nabla X^a = \tilde{D}X^a, \quad H \nabla X^\alpha = \tilde{D}X^\alpha, \quad V \nabla X^\alpha = \tilde{D}X^\alpha, \]

\( \omega^a_b, \omega^a_b, \omega^\alpha_\beta, \omega^\alpha_\beta, \omega^\alpha_\beta, \omega^\alpha_\beta \) are called the **connection 1-forms** of \( H \nabla \), \( V \nabla \).

### 4 The Gauss-Weingarten formulae

As usual in the theory of the submanifolds we are interested in finding the moving equations of the moving frame \( \mathcal{R} \) along \( \text{Osc}\tilde{M} \).

These equations, called also Gauss-Weingarten formulae, are obtained when the relative covariant derivatives of the vector fields from \( \mathcal{R} \) are expressed again in the frame \( \mathcal{R} \).

Thus we have

**Theorem 3.1** The following Gauss-Weingarten formulae hold:

\[ V_i \nabla B^a_\alpha = B^a_\sigma \Pi^\sigma_{\alpha}, \]

\[ V_i \nabla B^a_\alpha = -B^a_\sigma \Pi^\sigma_{\alpha}, \]
where
\[
V_i^\alpha = \begin{cases} 
V_i^\alpha & \text{if } i = 0 \\
\bar{H}_\alpha^\beta \delta \bar{u}^\beta + V_i^\beta & \text{if } i = 1
\end{cases}
\]
(25)
\[
\Pi^\alpha_\beta = g^{\alpha \sigma} \delta_{\delta \sigma} \Pi^\delta_\delta,
\]
and the d-tensors
\[
H^{\alpha \beta}_\delta = B^\delta_\delta \left( B^d_{\alpha \beta} + B^f_{\alpha} \bar{L}^d_{(0)} f^\beta \right)
\]
(26)
\[
H^{\alpha \beta}_\delta = B^\delta_\delta \left( B^d_{\alpha \beta} + B^f_{\alpha} V^d_{(1)} f^\beta \right)
\]
are the fundamental d-tensors of the second order of manifold \( \widetilde{\text{Osc}}M \), \((i = 0, 1, V_0 = H, V_1 = V)\).

**Proof** From (11) and (12) we have
\[
\nabla^H B^a_\alpha = B^a_{(0) \beta} du^\beta + B^a_{\alpha} |_{0 \beta} \delta v^\beta
\]
\[
= \left( \frac{\delta B^a_\alpha}{\delta u^\beta} + \bar{L}^a_{(00) \beta} - \frac{H^\delta}{(00)_{\alpha \beta}} B^a_\delta \right) du^\beta +
\]
\[
+ \left( \frac{\delta B^a_\alpha}{\delta v^\beta} + \bar{C}^a_{(01) \beta} - \frac{H^\delta}{(01)_{\alpha \beta}} B^a_\delta \right) \delta v^\beta
\]
\[
= B^a_{\alpha \beta} du^\beta + B^b_\alpha \left( \bar{L}^a_{(00) \beta} du^\beta + \bar{C}^a_{(01) \beta} \delta v^\beta \right) -
\]
\[
- B^a_\delta \left[ B^\delta_d \left( B^d_{\alpha \beta} + B^f_{\alpha} \bar{L}^d_{(00)} f^\beta \right) du^\beta + B^\delta_d B^f_{\alpha} \bar{C}^d_{(01) \beta} \delta v^\beta \right]
\]
Using (25) we get relation (23) for \( V_0 = H \).

Now, by applying \( \nabla^H \) to both sides of the equations
\[
g_{ab} B^a_\alpha B^b_\beta = 0
\]
one gets
\[
g_{ab} B^a_\delta \Pi^\delta_\alpha B^b_\beta + g_{ab} B^a_\alpha \Pi^b_\beta = 0.
\]
Multiplying these relation with \( B^a_d \) we obtain
\[
g_{ab} \nabla^H B^b_\beta - B^a_d B^\delta_d g_{ab} \nabla^H B^b_\beta = -B^a_d \delta_\beta_\gamma \Pi^\gamma_\alpha.
\]
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But \( B^a_{\beta} B^b_{\gamma} g_{ab} \nabla_{\beta} B^b_{\gamma} = 0 \). Consequently, we obtain the relations (24) for \( V_0 = H \).

Analogously, for the operator \( \nabla^V \) one gets the other relations.

References


