ON QUANTITATIVE ESTIMATION FOR THE LIMITING SEMIGROUP OF LINEAR POSITIVE OPERATORS

Bucurel MINEA

Abstract

We give an improved quantitative estimation of the approximation of iterates for a sequence of positive linear operators \((L_n)_n\) to their limiting \(C_0\)-semigroup \(T(t), 0 \leq t < \infty\), as given in Trotter’s theorem.

2010 Mathematics Subject Classification: 41A36.

Key words: positive linear operators, Trotter theorem, limiting semigroup of operators, Durrmeyer type operators

1 Introduction

The study of iterates of Bernstein operators was initiated by Kelisky and Rivlin [9] and the first quantitative results were be obtained by Nagel [11] and Gonska [6]. The semigroup of operators generated by iterates of Bernstein operators was considered by da Silva [5]. More generally, as application to the Trotter’s theorem [13] the semigroups generated by linear positive operators are considered in the last decades. For a general reference on semigroups of operators we cite [2] and [1]. Recently there were obtained quantitative results for Trotter’s theorem in certain contexts, see: [7], [10], [3], [8].

Our aim is to give a modified method then in the paper by Gonska and Raşa [7] for a quantitative version of the Trotter’s theorem in the case of a class of sequences of positive linear operators described below, which includes the sequence of Bernstein operators. We show that our method leads to estimates with better constants than in [7], when we apply this method to Bernstein operators.

Let \((L_n)_n : C[0,1] \rightarrow C^4[0,1]\) be a sequence of positive linear operators which preserves linear functions. We consider that \(L_n\) are convex of orders \(i\), for \(0 \leq i \leq 4\), i.e. if \(f \in C^i[0,1]\) and \(f^{(i)} \geq 0\) on \([0,1]\), then \((L_n(f))^{(i)} \geq 0\) on \([0,1]\). Denote

\[
M^k_n(x) := L_n((t-x)^k, x) \quad k \in \mathbb{N}, \ x \in [0,1].
\]

We suppose that \(M^2_n(x) = a_n x(1-x), x \in [0,1]\), where \(a_n = O\left(\frac{1}{n}\right), n \in \mathbb{N}\) and

\[
\lim_{n \to \infty} n(L_n(f, x) - f(x)) = \frac{a}{2} x(1-x) \cdot f''(x), \text{ uniformly, for } f \in C^2[0,1],
\]
where \( a = \lim_{n \to \infty} n a_n \).

We consider the differential operator \( A : D(A) = C^2[0,1] \to C[0,1] \), given by \( A(f)(x) = \frac{2}{3} x(1-x)f''(x) \), \( f \in C^2[0,1] \), \( x \in [0,1] \). Note that \( D(A) \) is dense in \( C[0,1] \).

In the condition above, from the general result of Trotter, we deduce that there exists a \( C_0 \)-semigroup \( T(t) \), such that
\[
\lim_{n \to \infty} L_n^m f = T(t)f, \ f \in C[0,1]
\] if \( m_n \to t, \ t \geq 0 \).

## 2 Main results

Firstly, we present two auxiliary general results.

**Lemma 1.** For every \( g \in C^4[0,1] \), we have:
\[
\| L_n g - g - \frac{1}{n} A g \| \leq \frac{1}{6} \| M_n^3 \| \| g^{(3)} \| + \frac{|a|}{a} \left| \frac{a}{n} - \frac{1}{n} \right| \cdot \| g'' \|.
\]

**Proof.** Let \( x \in [0,1] \) be fixed. For \( t \in [0,1] \), with Taylor’s formula, we write:
\[
\left| g(t) - g(x) - (t-x) g'(x) - \frac{1}{2} (t-x)^2 g''(x) \right| \leq \frac{1}{6} |t-x|^3 \| g^{(3)} \|.
\]

Since \( L_n \) reproduces linear functions we obtain:
\[
\| L_n g - g - \frac{a_n}{a} A g \| \leq \frac{1}{6} \| M_n^3 \| \| g^{(3)} \|,
\]
then we write
\[
\| L_n g - g - \frac{1}{n} A g \| \leq \| L_n g - g - \frac{a_n}{a} A g \| + \| \frac{a_n}{a} A g - \frac{1}{n} A g \|
\]
\[
\leq \frac{1}{6} \| M_n^3 \| \| g^{(3)} \| + \frac{|a|}{a} \left| \frac{a}{n} - \frac{1}{n} \right| \cdot \| A g \|
\]
\[
\leq \frac{1}{6} \| M_n^3 \| \| g^{(3)} \| + \frac{|a|}{8} \left| \frac{a}{n} - \frac{1}{n} \right| \cdot \| g'' \|.
\]

\qed

**Lemma 2.** For every \( g \in C^4[0,1] \), we have:
\[
\left\| T\left( \frac{1}{n} \right) g - g - \frac{1}{n} A g \right\| \leq \frac{a^2}{128n^2} (8 \| g'' \| + 8 \| g^{(3)} \| + \| g^{(4)} \|).
\]
Theorem 1. Let $f \in C^4[0, 1]$. The following estimation holds:

$$\|L_n^m f - T(t) f\| \leq \left[ \left( \frac{|a|}{8} \frac{a_n}{a} - \frac{1}{n} \right) + \frac{a^2}{16n^2} \frac{1 - (\sigma_2)^m}{1 - \sigma_2} + \frac{|a|}{8} \left( \frac{m}{n} - t \right) \right] \|f''\|
\quad + \left( \frac{1}{6} \|L_n^3 f\| + \frac{a^2}{16n^2} \frac{1 - (\sigma_3)^m}{1 - \sigma_3} \|f''\| \right) \|f^{(3)}\| + \frac{a^2}{128n^2} \frac{1 - (\sigma_4)^m}{1 - \sigma_4} \|f^{(4)}\|.$$ 

Proof. Firstly:

$$\left\|L_n^m f - T(t) f\right\| \leq \left\|L_n^m f - T\left( \frac{m}{n} \right) f\right\| + \left\|T\left( \frac{m}{n} \right) f - T(t) f\right\|
\leq \left\|L_n^m f - T\left( \frac{m}{n} \right) f\right\| + \left\| \int_t^{\frac{m}{n}} T(u)Af du \right\|
\leq \left\|L_n^m f - T\left( \frac{m}{n} \right) f\right\| + \frac{|a|}{8} \left( \frac{m}{n} - t \right) \|Af\|
\leq \left\|L_n^m f - T\left( \frac{m}{n} \right) f\right\| + \frac{|a|}{8} \left( \frac{m}{n} - t \right) \|f''\|. \quad (2)$$

Next, we present the main result of the paper.

**Lemma 3.** Let $f \in C^k[0, 1]$, $0 \leq k \leq 4$ and fix $n, j \geq 0$. The following estimation holds:

$$\|(L_n^j f)^{(k)}\| \leq (\sigma_k)^j \|f^{(k)}\|,$$

where $\sigma_k := \frac{1}{k!} (L_n e_k)^{(k)}$.

**Proof.** Denote $g := \frac{1}{k!} \|f^{(k)}\| e_k \pm f$. It is clear that $g^{(k)} \geq 0$. Since $L_n$ is convex of order $k$, we also have $(L_n g)^{(k)} \geq 0$, hence

$$\|L_n^{(k)} f\| \leq \frac{1}{k!} \|f^{(k)}\| (L_n e_k)^{(k)}$$

and by induction we obtain

$$\|(L_n^j f)^{(k)}\| \leq \left( \frac{1}{k!} (L_n e_k)^{(k)} \right)^j \|f^{(k)}\|.$$ 

\qed
Now, using a telescopic sum and \( \| T(t) \| = 1, \ t \geq 0 \), we write:

\[
\left\| L_mf - T\left( \frac{m}{n} \right) f \right\| = \left\| \sum_{j=0}^{m-1} T\left( \frac{m-1-j}{n} \right) \left( L_n - T\left( \frac{1}{n} \right) \right) L_n f \right\| 
\leq \sum_{j=0}^{m-1} \left\| \left( L_n - T\left( \frac{1}{n} \right) \right) L_n f \right\|. \tag{3}
\]

Denote \( g := L_n f \in C^4 [0,1] \). It is convenient to write:

\[
\left\| \left( L_n - T\left( \frac{1}{n} \right) \right) g \right\| \leq \left\| L_n g - g - \frac{1}{n} Ag \right\| + \left\| T\left( \frac{1}{n} \right) g - g - \frac{1}{n} Ag \right\|.
\]

Applying Lemmas 1 and 2, we can continue with

\[
\left\| \left( L_n - T\left( \frac{1}{n} \right) \right) g \right\| \leq \left( \frac{|a|}{8} \right) \frac{a_n}{a} - \frac{1}{n} \left\| g'' \right\|
+ \frac{a^2}{256n^2} \left( 8 \left\| g'' \right\| + 8 \left\| g^{(3)} \right\| + \left\| g^{(4)} \right\| \right)
= \left( \frac{|a|}{8} \right) \frac{a_n}{a} - \frac{1}{n} \left\| g'' \right\|
+ \left( \frac{1}{6} \right) \left\| M_3^n \right\| + \frac{a^2}{16n^2} \left\| g^{(3)} \right\|
+ \frac{a^2}{128n^2} \left\| g^{(4)} \right\|.
\]

Now, applying Lemma 3, we have \( \left\| g^{(j)} \right\| = (\sigma_k)^j \left\| f^{(j)} \right\|, \) for \( j = 2, 3, 4 \). Then, from (3)

\[
\left\| L_mf - T\left( \frac{m}{n} \right) f \right\| \leq \left( \frac{|a|}{8} \right) \frac{a_n}{a} - \frac{1}{n} \left\| f'' \right\|
+ \left( \frac{1}{6} \right) \left\| M_3^n \right\|
+ \frac{a^2}{16n^2} \frac{1 - \sigma_2^n}{1 - \sigma_3} \left\| f^{(3)} \right\|
+ \frac{a^2}{128n^2} \frac{1 - \sigma_4^n}{1 - \sigma_4} \left\| f^{(4)} \right\|. \tag{4}
\]

Combining (2) and (4) we now get the conclusion.

**Remark 1.** From the estimate given in Theorem 1 one could obtain estimates with moduli of smoothness, similarly as in [7]. See the method given there.

\[\blacksquare\]

### 3 Applications

Denote

\[ p_{nk}(x) := \binom{n}{k} x^k (1 - x)^{n-k}, \ x \in [0,1]. \]

I. Consider the sequence of Bernstein operators \( (B_n)_n \) given by

\[ B_n(f, x) = \sum_{k=0}^{n} p_{nk}(x) f \left( \frac{k}{n} \right), \ f \in C[0,1]. \]
For these operators, we have \( a = 1, a_n = \frac{1}{n}, \| M_n^2 \| = \frac{1}{4n^2} \) and, from [4] we deduce:

\[
\begin{align*}
\sigma_2 &= \frac{n-1}{n} \\
\sigma_3 &= \frac{(n-1)(n-2)}{n^2} \\
\sigma_4 &= \frac{(n-1)(n-2)(n-3)}{n^3}.
\end{align*}
\]

Hence, we find the following estimation:

\[
\| B_m^n f - T(t) f \| \leq \left( \frac{1 - (\sigma_2)^m}{16n} + \frac{1}{8} \left| \frac{m}{n} - t \right| \right) \| f'' \| + \frac{5(1 - (\sigma_2)^m)}{48(3n-2)} \| f^{(3)} \|
\]

\[
+ \frac{n(1 - (\sigma_4)^m)}{128(6n^2 - 11n + 6)} \| f^{(4)} \|.
\]

As consequence, we obtain:

\[
\| B_m^n f - T(t) f \| \leq \left( \frac{1}{16n} + \frac{1}{8} \left| \frac{m}{n} - t \right| \right) \| f'' \| + \frac{5}{48(3n-2)} \| f^{(3)} \|
\]

\[
+ \frac{n}{128(6n^2 - 11n + 6)} \| f^{(4)} \|.
\]

These estimate improves the following estimate given in [7] - formula (12):

\[
\| B_m^n f - T(t) f \| \leq \left( \frac{1}{8} \left| \frac{m}{n} - t \right| + \frac{1}{2n-1} \right) \| f'' \| + \frac{5}{96(2n-1)} \left( 4 \| f^{(3)} \| + \| f^{(4)} \| \right).
\]

II. For the sequence of genuine Durrmeyer operators \((U_n)_n\), given by

\[
U_n(f, x) = (1 - x)^n f(0) + x^n f(1) + (n-1) \sum_{k=1}^n p_{nk}(x) \int_0^1 p_{n-2,k-1}(t) f(t) dt
\]

we find \( a = 1, a_n = \frac{2}{2n+1}, \| M_n^3 \| = \frac{3}{2(2n+1)(2n+2)} \) and from [12], pag. 149 we deduce:

\[
\begin{align*}
\sigma_2 &= \frac{n-1}{n+1} \\
\sigma_3 &= \frac{(n-1)(n-2)}{(n+1)(n+2)} \\
\sigma_4 &= \frac{(n-1)(n-2)(n-3)}{(n+1)(n+2)(n+3)}.
\end{align*}
\]

Thus, we can write the following estimation:

\[
\| U_m^n f - T(t) f \| \leq \left[ (n+1) \left( \frac{1}{16n(2n+1)} + \frac{1}{32n^2} \right) + \frac{1}{8} \left| \frac{m}{n} - t \right| \right] \| f'' \|
\]

\[
+ \left( \frac{1}{4(2n+1)(2n+2)} + \frac{1}{16n^2} \right) \frac{n^2 + 3n + 2}{6n} \| f^{(3)} \|
\]

\[
+ \frac{n^3 + 6n^2 + 11n + 6}{1536n^2(n^2 + 1)} \| f^{(4)} \|.
\]
References


