A DIRECT APPROACH FOR TIME OPTIMAL CONTROL PROBLEM WITH LINEAR DIFFERENTIAL SYSTEM

Ernest SCHEIBER

Abstract

The purpose of this paper is to present an approach to solve the time-optimal control problem. While searching the control as a piecewise constant function the optimal control problem is reduced to a nonlinear programming problem. Two examples are presented, in which cases the computation is carried out with the Mathematica software.

2000 Mathematics Subject Classification: 49M25, 49M37.
Key words: time optimal control problem, nonlinear programming, computer algebra system.

1 Introduction

The control of a time optimal control problem with linear differential system is of bang-bang type with unknown switching times.

The time optimal control problem considered in this paper is

minimize $T$

subject to the constrains:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$x(0) = x_0$$
$$x(T) = x_T$$
$$u(t) \in U$$

where $x(t) \in \mathbb{R}^m$ and $u(t) \in \mathbb{R}^q$. The elements of the matrix $A(t) \in M_{m,m}(\mathbb{R})$ and $B(t) \in M_{m,q}(\mathbb{R})$ are supposed to be continuous. $U$ is a convex subset of $\mathbb{R}^q$.

Several computational techniques are derived to solve the time optimal control problems. These techniques are based on transforming into an optimization problem [1], [3] (i.e. the control functions and/or state functions are discretized) and/or transforming into
a two boundary value problem (the transformation is based on the necessary optimality conditions), [2].

Here we shall discretize only the control functions as in [2]. A control function will be searched as a piecewise constant function. In the given examples, using the Mathematica Computer Algebra System, the ordinary differential equations are integrated symbolically.

The main component of this approach is the minimization procedure.

2 The transformation of optimal control into nonlinear programming problem

Let $X(t)$ be a fundamental system of the linear differential system (2) (i.e. the columns of $X(t)$ are $m$ linear independent solutions of the homogeneous linear differential system $\dot{x}(t) = A(t)x(t)$). Denoting $H(t, s) = X(t)X^{-1}(s)$ the solution of the initial value problem (2)-(3) is

$$x(t) = H(t, 0)x_0 + \int_0^t H(t, s)B(s)u(s)ds.$$  

The constraint (4) will be

$$x_T = H(T, 0)x_0 + \int_0^T H(T, s)B(s)u(s)ds.$$  

or

$$\int_0^T X^{-1}(s)B(s)u(s)ds = X^{-1}(T)x_T - X^{-1}(0)x_0.$$  

(6)

We search an approximation the optimal control as a piecewise constant function: for a mesh

$$0 = t_0 < t_1 < \ldots < t_n = T$$

let be $\tilde{u}(t) = u_i, \ t \in (t_{i-1}, t_i], \ u_i \in U, \ i \in \{1, 2, \ldots, n\}$. Substituting $u = \tilde{u}$ in (6) we obtain

$$\sum_{i=1}^n \left( \int_{t_{i-1}}^{t_i} X^{-1}(s)B(s)ds \right) u_i = X^{-1}(T)x_T - X^{-1}(0)x_0.$$  

Denoting

$$C_i = \int_{t_{i-1}}^{t_i} X^{-1}(s)B(s)ds \in M_{m,q}(\mathbb{R}), \ i \in \{1, 2, \ldots, n\}$$

and

$$d = X^{-1}(T)x_T - X^{-1}(0)x_0 \in \mathbb{R}^m,$$

the following nonlinear programming problem

$$\text{minimize } g_0(T, u_1, \ldots, u_n) = T$$  

(7)
subject to

\[ g(T, u_1, \ldots, u_n) = \sum_{i=1}^{n} C_i u_i = d; \quad (8) \]

\[ u_i \in U \quad i \in \{1, 2, \ldots, n\}. \quad (9) \]

Usually, the nodes \( t_i \) are equidistant, \( t_i = \frac{T}{n} i, \quad i \in \{1, 2, \ldots, n\} \), for some prescribed \( n \in \mathbb{N}^* \). As a consequence matrix \( C_i \) depends on \( T \) and thus the minimization problem is nonlinear.

### 3 Examples

The computation was carried out with Mathematica. Mathematica allows a simple and nice way to generate the constraints for any \( n \in \mathbb{N}^* \). The minimization is realized calling the `NMinimize` Mathematica function.

1. minimize \( T \)

subject to

\[ \dot{x}_1 = x_2 \quad x_1(0) = x_1^0 \quad x_1(T) = 0 \]
\[ \dot{x}_2 = u \quad x_2(0) = x_2^0 \quad x_2(T) = 0 \]
\[ |u| \leq 1 \]

The solution may be easily computed using the Pontryagin’s maximum principle, \([4]\). Almost any introductory tutorial of optimal control presents this example, but here we are interested in a solution obtained by a computer for arbitrary \( x_1^0, x_2^0 \). The plot of the possible optimal trajectories a given in Fig.1.

![Fig. 1: The shape of the optimal state trajectories.](image-url)
A fundamental matrix of the corresponding homogeneous system is

\[
X(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.
\]

The constraints (8) are

\[
\begin{align*}
\sum_{i=1}^{n} u_i \int_{t_{i-1}}^{t_i} s ds &= x_1^0, \\
\sum_{i=1}^{n} u_i \int_{t_{i-1}}^{t_i} ds &= -x_2^0.
\end{align*}
\]

For equidistant nodes, the above equations become

\[
\begin{align*}
\frac{T^2}{n} \sum_{i=1}^{n} (i - \frac{1}{2}) u_i &= x_1^0 \iff \frac{T^2}{n} \sum_{i=1}^{n} i u_i = x_1^0 - \frac{T}{2n} x_2^0 \\
\frac{T}{n} \sum_{i=1}^{n} u_i &= -x_2^0
\end{align*}
\]

The optimization problem is: minimize \( T \) subject to the constraints:

\[
\begin{align*}
\frac{T^2}{n^2} \sum_{i=1}^{n} i u_i - x_1^0 &+ \frac{T}{2n} x_2^0 = 0, \\
\frac{T}{n} \sum_{i=1}^{n} u_i + x_2^0 &= 0, \\
|u_i| &\leq 1, \ \forall i \in \{1, 2, \ldots, n\}.
\end{align*}
\]

The Mathematica code to solve this nonlinear programming problem is

```
OCP[n_, x10_, x20_] :=
NMinimize[
Join[
{T, x10 - T x20/(2 n) -
T^2/n^2 Sum[
ToExpression[StringJoin["u", ToString[i]]], {i, 1, n}] == 0,
x20 + T/n Sum[
ToExpression[StringJoin["u", ToString[i]]], {i, 1, n}] == 0,
T > 0}, Table[
ToExpression[StringJoin["-1<=u", ToString[i]]], {i, 1, n}],
Table[ToExpression[StringJoin["t>=u", ToString[i]]], {i, 1, n}],
Join[{T}, Table[ToExpression[StringJoin["u", ToString[i]]], {i, 1, n}]]
],
{u1, u2, ..., u20}, {T}]]
```

To obtain a valid solution, there is required the additional constraint \( T > 0 \).

For \( n = 64 \), \( (x_1^0, x_2^0) = (3, 2) \in D^- \) we have obtained \( T = 6.47259 \) in concordance with the theoretical value \( T = x_2^0 + 2 \sqrt{\frac{1}{2}(x_2^0)^2 + x_1^0} \). The obtained results are

\{6.47259, {T -> 6.47259, u1 -> -1., u2 -> -1., u3 -> -1., u4 -> -1., u5 -> -1., u6 -> -1., u7 -> -1., u8 -> -1., u9 -> -1., u10 -> -1., u11 -> -1., u12 -> -1., u13 -> -1., u14 -> -1., u15 -> -1., u16 -> -1., u17 -> -1., u18 -> -1., u19 -> -1., u20 -> -1., u21 -> -1., u22 -> -1., u23 -> -1., u24 -> -1., u25 -> -1.\}
A Direct approach for time-optima control problem

\begin{align*}
\begin{array}{cccccccc}
  u_{26} & \rightarrow & -1., & u_{27} & \rightarrow & -1., & u_{28} & \rightarrow & -1., \\
  u_{31} & \rightarrow & -0.999999, & u_{32} & \rightarrow & -0.999999, & u_{33} & \rightarrow & -0.999999, \\
  u_{34} & \rightarrow & -0.999999, & u_{35} & \rightarrow & -0.999999, & u_{36} & \rightarrow & -0.999999, \\
  u_{37} & \rightarrow & -0.999999, & u_{38} & \rightarrow & -0.999999, & u_{39} & \rightarrow & -0.999999, \\
  u_{40} & \rightarrow & -0.9999997, & u_{41} & \rightarrow & -0.9999993, & u_{42} & \rightarrow & -0.775689, \\
  u_{43} & \rightarrow & 0.999999, & u_{44} & \rightarrow & 0.999996, & u_{45} & \rightarrow & 0.999998, \\
  u_{46} & \rightarrow & 0.999998, & u_{47} & \rightarrow & 0.999999, & u_{48} & \rightarrow & 0.999999, \\
  u_{49} & \rightarrow & 0.999999, & u_{50} & \rightarrow & 0.999999, & u_{51} & \rightarrow & 0.999999, \\
  u_{52} & \rightarrow & 0.999999, & u_{53} & \rightarrow & 0.999999, & u_{54} & \rightarrow & 1., \\
  u_{55} & \rightarrow & 1., & u_{56} & \rightarrow & 1., & u_{57} & \rightarrow & 1., & u_{58} & \rightarrow & 1., & u_{59} & \rightarrow & 1., & u_{60} & \rightarrow & 1., \\
  u_{61} & \rightarrow & 1., & u_{62} & \rightarrow & 1., & u_{63} & \rightarrow & 1., & u_{64} & \rightarrow & 1. \\
\end{array}
\end{align*}

The plot of the corresponding control is given in Fig. 2.

\begin{center}
\begin{tikzpicture}
\begin{axis}[
    width=\textwidth,
    xlabel={t},
    ylabel={u},
    ymin=-1.0, ymax=1.0,
    xmin=0, xmax=6
]
\addplot [domain=0:6, samples=100, smooth] {0};
\end{axis}
\end{tikzpicture}
\end{center}

Fig. 2: Plot of the control for Example 1.

An approximation of the switching time is $\xi_1 \approx t_{42} = 4.24764$.

2. The second problem is chosen from [2] (with several included references):

$$\text{minimize } T$$

subject to

\begin{align*}
\dot{x}_1 &= x_3 & x_1(0) &= 0 & x_1(T) &= 1 \\
\dot{x}_2 &= x_4 & x_2(0) &= 0 & x_2(T) &= 1 \\
\dot{x}_3 &= \frac{u}{m_1} - \frac{k}{m_1} (x_1 - x_2) & x_3(0) &= 0 & x_3(T) &= 0 \\
\dot{x}_4 &= \frac{k}{m_2} (x_1 - x_2) & x_4(0) &= 0 & x_4(T) &= 0 \\
\lvert u \rvert &\leq 1
\end{align*}

For $m_1 = m_2 = k = 1$, using Mathematica we have found the fundamental matrix

$$X(t) = \begin{pmatrix}
\cos \left( \frac{t}{\sqrt{2}} \right)^2 & \sin \left( \frac{t}{\sqrt{2}} \right)^2 & \frac{1}{4} \left( 2t + \sqrt{2} \sin \left( \sqrt{2}t \right) \right) & \frac{1}{4} \left( 2t - \sqrt{2} \sin \left( \sqrt{2}t \right) \right) \\
\sin \left( \frac{t}{\sqrt{2}} \right)^2 & \cos \left( \frac{t}{\sqrt{2}} \right)^2 & \frac{1}{4} \left( 2t - \sqrt{2} \sin \left( \sqrt{2}t \right) \right) & \frac{1}{4} \left( 2t + \sqrt{2} \sin \left( \sqrt{2}t \right) \right) \\
-\frac{\sin(\sqrt{2}t)}{\sqrt{2}} & \frac{\sin(\sqrt{2}t)}{\sqrt{2}} & \cos \left( \frac{t}{\sqrt{2}} \right)^2 & \sin \left( \frac{t}{\sqrt{2}} \right)^2 \\
\frac{\sin(\sqrt{2}t)}{\sqrt{2}} & -\frac{\sin(\sqrt{2}t)}{\sqrt{2}} & -\sin \left( \frac{t}{\sqrt{2}} \right)^2 & \cos \left( \frac{t}{\sqrt{2}} \right)^2
\end{pmatrix}$$
with the inverse

\[ X^{-1}(t) = \]

\[
\begin{pmatrix}
\cos \left( \frac{t}{\sqrt{2}} \right)^2 & \sin \left( \frac{t}{\sqrt{2}} \right)^2 & \frac{1}{4} (-2t - \sqrt{2}\sin [\sqrt{2} t]) & \frac{1}{4} (-2t + \sqrt{2}\sin [\sqrt{2} t]) \\
\sin \left( \frac{t}{\sqrt{2}} \right)^2 & \cos \left( \frac{t}{\sqrt{2}} \right)^2 & \frac{1}{4} (-2t + \sqrt{2}\sin [\sqrt{2} t]) & \frac{1}{4} (-2t - \sqrt{2}\sin [\sqrt{2} t]) \\
\frac{\sin [\sqrt{2} t]}{\sqrt{2}} & \frac{-\sin [\sqrt{2} t]}{\sqrt{2}} & \cos \left( \frac{t}{\sqrt{2}} \right)^2 & \sin \left( \frac{t}{\sqrt{2}} \right)^2 \\
\frac{-\sin [\sqrt{2} t]}{\sqrt{2}} & \frac{\sin [\sqrt{2} t]}{\sqrt{2}} & \sin \left( \frac{t}{\sqrt{2}} \right)^2 & \cos \left( \frac{t}{\sqrt{2}} \right)^2 
\end{pmatrix}
\]

Because \( \int X^{-1}(t)B(s)ds = \)

\[
\left\{ \frac{1}{4} \left(-s^2 + \cos \left[ \sqrt{2} s \right]\right), \frac{1}{4} \left(-s^2 - \cos \left[ \sqrt{2} s \right]\right), s + \frac{\sin \left[ \sqrt{2} s \right]}{2}, \frac{s - \sin \left[ \sqrt{2} s \right]}{2} \right\}
\]

and \( d = \{1, 1, 0, 0\} \), after some simple algebraic processing, the constraints (8) become

\[
\frac{T^2}{n^2} \sum_{i=1}^{n} i u_i = -1, \quad \sum_{i=1}^{n} u_i \cos \frac{T(i-\frac{1}{2})\sqrt{2}}{n} = 0, \\
\sum_{i=1}^{n} u_i \sin \frac{T(i-\frac{1}{2})\sqrt{2}}{n} = 0, \quad \sum_{i=1}^{n} u_i = 0.
\]

The Mathematica code for the minimization function is

```mathematica
OCP[un_] :=
  NMinimize[
    Join[{T, T^2/(2 n^2) Sum[
        i ToExpression[StringJoin["u", ToString[i]]], {i, 1, n}] == -1,
     Sum[ToExpression[StringJoin["u", ToString[i]]], {i, 1, n}] == 0,
     Sum[ToExpression[StringJoin["u", ToString[i]]] Cos[
        T (1 - 0.5) Sqrt[2]/n], {i, 1, n}] == 0,
     Sum[ToExpression[StringJoin["u", ToString[i]]] Sin[
        T (1 - 0.5) Sqrt[2]/n], {i, 1, n}] == 0, T > 3},
     Table[ToExpression[StringJoin["-1<i<="u", ToString[i]]], {i, 1, n}],
     Table[ToExpression[StringJoin["1>="u", ToString[i]]], {i, 1, n}],
     Join[{T},
     Table[ToExpression[StringJoin["u", ToString[i]]], {i, 1, n}]]]],
  {{T, 0.1}, {n, 10}}]
```

Here the additional constraint is \( T > 3 \).

For \( n = 32 \) we obtained

\{4.22218, \{T -> 4.22218, u1 -> 1., u2 -> 1., u3 -> 1., u4 -> 1.,
  u5 -> 1., u6 -> 1., u7 -> 1., u8 -> 0.228417, u9 -> -1., u10 -> -1.,
  u11 -> -1., u12 -> -1., u13 -> -1., u14 -> -1., u15 -> -1.,
  u16 -> -1., u17 -> 1., u18 -> 1., u19 -> 1., u20 -> 1., u21 -> -1.,
  u22 -> 1., u23 -> 1., u24 -> 1., u25 -> -0.228417, u26 -> -1.,
  u27 -> -1., u28 -> -1., u29 -> -1., u30 -> -1., u31 -> -1.,
  u32 -> -1.\}]

Thus \( T = 4.22218 \). The plot of the corresponding control is given in Fig. 3.

The approximation of the switching times is \( \xi_1 \approx t_8 = 1.05555, \xi_2 \approx t_{16 + \frac{1}{2}} = 2.17706, \xi_3 \approx t_{25} = 3.29858 \). These results agree with the results reported in [2].

The drawback of this approach is that an additional constraint is required and that the time to evaluate the minimization function is frustrating. On a two cores computer, the duration to solve the two examples, for \( n = 64 \) and respectively \( n = 32 \), is a few minutes.
4 Conclusions

If the transformation of the optimal control problem into a mathematical programming problem is straightforward, the contribution of this paper is the Mathematica coding to generate that mathematical programming problem. It results a simple method to solve a class of time optimal control problems. The method requires only the general-purpose Mathematica software.

References


