A GENERALIZATION OF KANTOROVICH OPERATORS AND A SHAPE-PRESERVING PROPERTY OF BERNSTEIN OPERATORS

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Abstract

We construct a generalization of the Kantorovich operators, depending on a parameter $b \geq 0$ and we prove that if a function $f \in C^1[0,1]$ with $f(0) = 0$, satisfies the differential inequality $f' + bf \geq 0$, then functions $B_n(f), n \in \mathbb{N}$ satisfy the same inequality, where $B_n$ are the Bernstein operators.

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1 Introduction

The Bernstein operators on the space $C[0,1]$ are defined by:

$$B_n(f, x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)p_{n,k}(x), \quad f \in C[a,b], \quad x \in [0,1], \quad n \in \mathbb{N},$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$ (1)

The Kantorovich modification of the Bernstein operators are given by:

$$K_n(f, x) = (n + 1) \sum_{k=0}^{n} p_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt, \quad f \in C[0,1], \quad x \in [0,1], \quad n \in \mathbb{N}.$$ (2)

We note that, the Kantorovich operators $K_n$ can be obtained by the following formula

$$K_n = D \circ B_{n+1} \circ I,$$ (3)

where $D$ is the differentiation operator: $D(f) = f', \quad f \in C_1[0,1]$ and $I$ is the antiderivative operator: $I(f, x) = \int_{0}^{x} f(t)dt, \quad f \in C[0,1], \quad x \in [0,1]$. More general, if $L : C[0,1] \to C^r[0,1]$ is an arbitrary linear operator and $r \in \mathbb{N}$, if we denote by $D^r$ and $I^r$, the iterates of operators $D$ and $I$, then the operator $D^r \circ L \circ I^r$ is named the Kantorovich modification of operator $L$ of order $r$. These operators play a crucial role in simultaneous approximation. Other types of generalizations or modifications of Kantorovich operators, partially included in References, are also known.

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2 Definition. Main results

We consider a generalization of the Kantorovich operators in the following sense.

**Definition 2.1.** Let a parameter $b \geq 0$. For any $n \in \mathbb{N}$ define the operator $K_n^b : C[0, 1] \to C[0, 1]$, defined by

$$K_n^b(f, x) := (n + 1 + b) \sum_{k=0}^{n} p_{n,k}(x) e^{-b \frac{k+1}{n+1}} \int_0^{\frac{k}{n+1}} e^{bt} f(t) dt$$

$$+ \sum_{k=0}^{n} p_{n,k}(x) [(n + 1 + b) - (n + 1 - b)e^{\frac{k}{n+1}}] e^{-b \frac{k+1}{n+1}} \int_0^{\frac{k}{n+1}} e^{bt} f(t) dt, \quad (4)$$

for $f \in C[0, 1], \ x \in [0, 1]$.

**Remark 2.1.** If we take $b = 0$ in (4) we obtain the Kantorovich operators given in (2).

**Theorem 2.1.** Operators $K_n^b$ are linear and positive, for any $n \in \mathbb{N}$ and $b \geq 0$.

**Proof.** The linearity is clear. In order to prove the positivity it is enough to show that

$$(n + 1 + b) - (n + 1 - b)e^{\frac{b}{n+1}} \geq 0.$$  

Consider function $\varphi(t) = 1 + t + (t - 1)e^t, \ t \in \mathbb{R}$. If we denote $t = \frac{b}{n+1}$ it is sufficient to show that $\varphi(t) \geq 0$, for $t \geq 0$. We have $\varphi'(t) = 1 + te^t$. The minimum of function $\varphi'$ is reached at point $t = -1$ and $\varphi'(-1) = 1 - e^{-1} > 0$. Hence $\varphi'(t) > 0, \ t \in \mathbb{R}$. Then function $\varphi$ is increasing on $\mathbb{R}$. But $\varphi(0) = 0$ and hence $\varphi(t) \geq 0$, for $t \geq 0$.

In order to give another description of operators $K_n^b$ we consider operators $D_b : C^1[0, 1] \to C^1[0, 1]$ and $I_b : C[0, 1] \to C^1[0, 1]$, given by

$$D_b(f, x) = f'(x) + bf(x), \ f \in C^1[0, 1], \ x \in [0, 1],$$

$$I_b(f, x) = e^{-bx} \int_0^x e^{bt} f(t) dt, \ f \in C[0, 1], \ x \in [0, 1].$$

**Lemma 2.1.** Let $n \in \mathbb{N}$ and $b \geq 0$. We have

i) $(D_b \circ I_b)(f) = f$, for all $f \in C[0, 1]$,

ii) $(I_b \circ D_b)(f) = f$, for all $f \in C^1[0, 1], \ such \ that \ f(0) = 0$.

**Proof.** i) If $f \in C[0, 1]$, then $I_b(f)$ is the solution of the Cauchy problem $y' + by = f$, $y(0) = 0$. Then $(D_b \circ I_b)(f) = f$.

ii) If $f \in C^1[0, 1]$ and $f(0) = 0$, then integrating by parts we obtain, for $x \in [0, 1]$:

$$(I_b \circ D_b)(f, x) = e^{-bx} \int_0^x e^{bt}(f'(t) + bf(t)) dt$$

$$= e^{-bx} \left[ e^{bx} f(x) - f(0) - b \int_0^x e^{bt} f(t) dt + b \int_0^x e^{bt} f(t) dt \right]$$

$$= f(x).$$
Theorem 2.2. For any \( n \in \mathbb{N} \) and \( b \geq 0 \) we have:

\[
K_n^b = D_b \circ B_{n+1} \circ I_b. 
\]  

(5)

Proof. Let \( f \in C[0,1] \) and \( x \in [0,1] \). Using the convention \( P_{n,k}(x) = 0 \), for \( k < 0 \) or \( k > n \), we have:

\[
(D_b \circ B_{n+1} \circ I_b)(f, x) = (B_{n+1}(I_b(f), x))' + bB_{n+1}(I_b(f), x)
\]

\[
= (n + 1) \sum_{k=0}^{n+1} [p_{n,k-1}(x) - p_{n,k}(x)] I_b\left(\frac{k}{n+1}\right)
\]

\[
b \sum_{k=0}^{n+1} [p_{n,k-1}(x) + p_{n,k}(x)] I_b\left(\frac{k}{n+1}\right)
\]

\[
= \sum_{k=0}^{n+1} [(n + 1 + b)p_{n,k-1}(x) - (n + 1 - b)p_{n,k}(x)] I_b\left(\frac{k}{n+1}\right)
\]

\[
= \sum_{k=0}^{n} p_{n,k}(x) \left[ (n + 1 + b) I_b\left(\frac{k + 1}{n+1}\right) - (n + 1 - b) I_b\left(\frac{k}{n+1}\right) \right].
\]

From this it follows immediately (4). \( \square \)

The results above allow us to derive a more general shape-preservation property for Bernstein operators. For this, let \( b \geq 0 \). Set

\[
D_b := \{ f \in C^1[0,1] : D_b(f) \geq 0, f(0) = 0 \}.
\]  

(6)

We have

Theorem 2.3. For any \( n \in \mathbb{N} \), \( n \geq 2 \) and \( b \geq 0 \), we have \( B_n(D_b) \subset D_b \).

Proof. Let \( f \in D_b \). We have \((D_b \circ B_n)(f) = (D_b \circ B_n \circ I_b)(D_b(f)) = K_{n-1}^b(D_b(f))\). Since \( D_b(f) \geq 0 \) and \( K_{n-1}^b \) is a positive operator it follows \( K_{n-1}^b(D_b(f)) \geq 0 \), i.e. \((D_b \circ B_n)(f) \geq 0\). Also \( B_n(f, 0) = f(0) = 0 \). Hence \( B_n(f) \in D_b \). \( \square \)

Theorem 2.4. We have

\[
K_n^b(f) \Rightarrow f
\]  

(7)

for all \( f \in C[0,1] \).

(The symbol \( \Rightarrow \) means the uniform convergence on the interval \([0,1]\).)

Proof. Since operators \( K_n^b \) are positive it suffices to prove relation (7) for three test functions. Let us denote \( e_k(t) = t^k, t \in [0,1], \) for \( k = 0, 1, 2 \). Then denote \( g_k = I_b(e_k), k = 0, 1, 2 \). From the convergence properties of Bernstein operators we have \( B_{n+1}(g_k) \Rightarrow g_k \) and \( (B_{n+1}(g_k))' \Rightarrow g_k \), for \( k = 0, 1, 2 \). Hence, for the same indices \( k \) we have \((D_b \circ B_{n+1})(g_k) \Rightarrow D_b(g_k)\). But \((D_b \circ B_{n+1})(g_k) = K_n^b(e_k)\) and \( D_b(g_k) = e_k \). Hence \( K_n^b(e_k) \Rightarrow e_k \), for \( k = 0, 1, 2 \). Therefore we can apply the theorem of Popoviciu-Bohmann-Korovkin and we obtain (7). \( \square \)
References


Erratum


The author