EXISTENCE OF POSITIVE SOLUTIONS FOR A NONLINEAR HIGHER-ORDER MULTI-POINT BOUNDARY VALUE PROBLEM

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Abstract

We investigate the existence of positive solutions of a system of higher-order nonlinear ordinary differential equations, subject to multi-point boundary conditions.

2000 Mathematics Subject Classification: 34B10, 34B18.

Key words: higher-order differential system, multi-point boundary conditions, positive solutions.

1 Introduction

In recent years, the multi-point boundary value problems for second-order or higher-order differential or difference equations/systems have been investigated by many authors, by using different methods such us fixed point theorems in cones, the Leray-Schauder continuation theorem and its nonlinear alternatives and the coincidence degree theory.

In this paper, we consider the system of nonlinear higher-order ordinary differential equations

\[(S) \quad \begin{cases} u^{(n)}(t) + \lambda c(t)f(u(t), v(t)) = 0, & t \in (0, T), \ n \in \mathbb{N}, \ n \geq 2, \\ v^{(m)}(t) + \mu d(t)g(u(t), v(t)) = 0, & t \in (0, T), \ m \in \mathbb{N}, \ m \geq 2, \end{cases}\]

with the multi-point boundary conditions

\[(BC) \quad \begin{cases} u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \ u(T) = \sum_{i=1}^{p-2} a_i u(\xi_i), \ p \in \mathbb{N}, \ p \geq 3, \\ v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0, \ v(T) = \sum_{i=1}^{q-2} b_i v(\eta_i), \ q \in \mathbb{N}, \ q \geq 3. \end{cases}\]

We give sufficient conditions on \(\lambda, \mu, f\) and \(g\) such that positive solutions of \((S)-(BC)\) exist. By a positive solution of problem \((S)-(BC)\) we mean a pair of functions \((u, v) \in \)

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Theorem 1. Let \((X, \|\cdot\|)\) be a normed linear space, \(K \subset X\) a cone, \(0 < a < b\) two given numbers and \(K(a, b) = \{x \in K, a \leq \|x\| \leq b\}\), \(K_a = \{x \in K, \|x\| = a\}\), \(K_b = \{x \in K, \|x\| = b\}\). Let \(T : K(a, b) \to K\) be a completely continuous operator such that one of the following conditions is satisfied:

i) \(\|Tx\| \leq \|x\|\) if \(x \in K_a\) and \(\|Tx\| \geq \|x\|\) if \(x \in K_b\);

ii) \(\|Tx\| \geq \|x\|\) if \(x \in K_a\) and \(\|Tx\| \leq \|x\|\) if \(x \in K_b\).

Then \(T\) has a fixed point in \(K(a, b)\).

Finally, some examples are presented in Section 4 to illustrate our main results.

2 Auxiliary results

In this section, we present some auxiliary results from [11] and [16], related to the following \(n\)-order differential equation with \(p\)-point boundary conditions

\[ u^{(n)}(t) + g(t) = 0, \quad t \in (0, T), \]

\[ n \in \mathbb{N}, \quad p \in \mathbb{N}, \quad T > 0, \quad g \in C^0((0, T), \mathbb{R}). \]
Lemma 1. ([11], [16]) If \( d = T^{n-1} - \sum_{i=1}^{p-2} a_i \xi_i^{n-1} \neq 0 \), \( 0 < \xi_1 < \cdots < \xi_{p-2} < T \) and \( y \in C([0,T]) \), then the solution of (1)-(2) is given by

\[
\begin{align*}
    u(t) &= \frac{t^{n-1}}{d(n-1)!} \int_0^T (T-s)^{n-1} y(s) \, ds - \frac{t^{n-1}}{d(n-1)!} \sum_{i=1}^{p-2} a_i \int_0^T (\xi_i - s)^{n-1} y(s) \, ds \\
    &\quad - \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} y(s) \, ds, \quad 0 \leq t \leq T.
\end{align*}
\]

Lemma 2. ([11], [16]) Under the assumptions of Lemma 1, the Green’s function for the boundary value problem (1)-(2) is given by

\[
G_1(t,s) = \begin{cases} 
    \frac{t^{n-1}}{d(n-1)!} \left( (T-s)^{n-1} - \sum_{i=j+1}^{p-2} a_i (\xi_i - s)^{n-1} \right) - \frac{1}{(n-1)!} (t-s)^{n-1}, & \text{if } \xi_j \leq s < \xi_{j+1}, \: s \leq t, \\
    \frac{t^{n-1}}{d(n-1)!} \left( (T-s)^{n-1} - \sum_{i=j+1}^{p-2} a_i (\xi_i - s)^{n-1} \right), & \text{if } \xi_j \leq s < \xi_{j+1}, \: s \geq t, \: j = 0, \ldots, p-3, \\
    \frac{t^{n-1}}{d(n-1)!} (T-s)^{n-1} - \frac{1}{(n-1)!} (t-s)^{n-1}, & \text{if } \xi_{p-2} \leq s \leq T, \: s \leq t, \\
    \frac{t^{n-1}}{d(n-1)!} (T-s)^{n-1}, & \text{if } \xi_{p-2} \leq s \leq T, \: s \geq t, \: (\xi_0 = 0).
\end{cases}
\]

Using the above Green’s function the solution of problem (1)-(2) is expressed as

\[
u(t) = \int_0^T G_1(t,s)y(s) \, ds.
\]

Lemma 3. ([11], [16]) If \( a_i > 0 \) for all \( i = 1, \ldots, p-2 \), \( 0 < \xi_1 < \cdots < \xi_{p-2} < T \), \( d > 0 \) and \( y \in C([0,T]) \), \( y(t) \geq 0 \) for all \( t \in [0,T] \), then the solution \( u(t) \) of problem (1)-(2) satisfies \( u(t) \geq 0 \) for all \( t \in [0,T] \).

Lemma 4. ([16]) If \( a_i > 0 \) for all \( i = 1, \ldots, p-2 \), \( 0 < \xi_1 < \cdots < \xi_{p-2} < T \), \( d > 0 \), \( y \in C([0,T]) \), \( y(t) \geq 0 \) for all \( t \in [0,T] \), then the solution of problem (1)-(2) satisfies

\[
\begin{align*}
    u(t) &\leq \frac{T^{n-1}}{d(n-1)!} \int_0^T (T-s)^{n-1} y(s) \, ds, \quad \forall t \in [0,T], \\
    u(\xi_j) &\geq \frac{\xi_j^{n-1}}{d(n-1)!} \int_{\xi_{p-2}}^T (T-s)^{n-1} y(s) \, ds, \quad \forall j = 1, p-2.
\end{align*}
\]
Lemma 5. ([11]) Assume that $0 < \xi_1 < \cdots < \xi_{p-2} < T$, $a_i > 0$ for all $i = 1, \ldots, p-2$, $d > 0$ and $y \in C([0, T])$, $y(t) \geq 0$ for all $t \in [0, T]$. Then the solution of problem (1)-(2) satisfies 
$$\inf_{t \in [\xi_{p-2}, T]} u(t) \geq \gamma_1 \|u\|,$$ where
\[
\gamma_1 = \begin{cases}
\min \left\{ \frac{a_{p-2}(T - \xi_{p-2})}{T - a_{p-2}\xi_{p-2}}, \frac{a_{p-2}\xi_{p-2}^{n-1}}{T^{n-1}} \right\}, & \text{if } \sum_{i=1}^{p-2} a_i < 1, \\
\min \left\{ \frac{a_i \xi_{i}^{n-1}}{T^{n-1}}, \frac{\xi_{i}^{n-1}}{T^{n-1}} \right\}, & \text{if } \sum_{i=1}^{p-2} a_i \geq 1.
\end{cases}
\]

We can also formulate similar results as Lemma 1 - Lemma 5 above for the boundary value problem
\[
v^{(m)}(t) + h(t) = 0, \ t \in (0, T),
\]
\[
v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0, \ v(T) = \sum_{i=1}^{q-2} b_i v(\eta_i).
\]

If $e = T^{m-1} - \sum_{i=1}^{q-2} b_i \eta_i^{m-1} \neq 0$, $0 < \eta_1 < \cdots < \eta_{q-2} < T$ and $h \in C([0, T])$, we denote by $G_2$ the Green’s function corresponding to problem (3)-(4). Under similar assumptions as those from Lemma 5, we have the inequality $\inf_{t \in [\eta_{q-2}, T]} v(t) \geq \gamma_2 \|v\|$, where $v$ is the solution of problem (3)-(4) and $\gamma_2$ has a similar form as $\gamma_1$ from Lemma 5 with $n$, $p$ and $a_i$ replaced by $m$, $q$ and $b_i$, respectively.

3 Main results

In this section, we give sufficient conditions on $\lambda$, $\mu$, $f$ and $g$ such that positive solutions with respect to a cone for our problem $(S) - (BC)$ exist.

We present the assumptions that we shall use in the sequel.

(H1) $0 < \xi_1 < \cdots < \xi_{p-2} < T$, $a_i > 0$, $i = 1, p - 2$, $d = T^{n-1} - \sum_{i=1}^{p-2} a_i \xi_i^{n-1} > 0$,
$0 < \eta_1 < \cdots < \eta_{q-2} < T$, $b_i > 0$, $i = 1, q - 2$, $e = T^{m-1} - \sum_{i=1}^{q-2} b_i \eta_i^{m-1} > 0$.

(H2) The functions $c$, $d : [0, T] \to [0, \infty)$ are continuous and there exist $t_1$, $t_2 \in [\theta_0, T]$ such that $c(t_1) > 0$ and $d(t_2) > 0$, where $\theta_0 = \max\{\xi_{p-2}, \eta_{q-2}\}$.

(H2') The functions $c$, $d : [0, T] \to [0, \infty)$ are continuous and there exist $t_1 \in [\xi_{p-2}, T]$, $t_2 \in [\eta_{q-2}, T]$ such that $c(t_1) > 0$ and $d(t_2) > 0$.

(H3) The functions $f$, $g : [0, \infty) \times [0, \infty) \to [0, \infty)$ are continuous.
Throughout this section, we let
\[
\begin{align*}
f_0^s &= \limsup_{(u,v) \to (0^+,0^+)} \frac{f(u,v)}{u + v}, \quad g_0^s &= \limsup_{(u,v) \to (0^+,0^+)} \frac{g(u,v)}{u + v}, \\
f_0^i &= \liminf_{(u,v) \to (0^+,0^+)} \frac{f(u,v)}{u + v}, \quad g_0^i &= \liminf_{(u,v) \to (0^+,0^+)} \frac{g(u,v)}{u + v}, \\
f_\infty^s &= \limsup_{(u,v) \to (\infty,\infty)} \frac{f(u,v)}{u + v}, \quad g_\infty^s &= \limsup_{(u,v) \to (\infty,\infty)} \frac{g(u,v)}{u + v}, \\
f_\infty^i &= \liminf_{(u,v) \to (\infty,\infty)} \frac{f(u,v)}{u + v}, \quad g_\infty^i &= \liminf_{(u,v) \to (\infty,\infty)} \frac{g(u,v)}{u + v}.
\end{align*}
\]

We consider the Banach space \( X = C([0,T]) \) with supremum norm \( \| \cdot \| \), and the Banach space \( Y = X \times X \) with the norm \( \|(u,v)\|_Y = \|u\| + \|v\| \).

We define the cone \( C \subset Y \) by
\[
C = \{(u,v) \in Y; \ u(t) \geq 0, \ v(t) \geq 0, \ \forall t \in [0,T] \ \text{and} \ \inf_{t \in [0,T]} (u(t) + v(t)) \geq \gamma \|(u,v)\|_Y \},
\]
where \( \gamma = \min\{\gamma_1, \gamma_2\} \) and \( \gamma_1, \gamma_2 \) are defined in Section 2.

First, for \( f_0^s, g_0^s, f_\infty^s, g_\infty^s \in (0, \infty) \) and positive numbers \( \alpha_1, \alpha_2 > 0 \) such that \( \alpha_1 + \alpha_2 = 1 \), we define the positive numbers \( L_1, L_2, L_3 \) and \( L_4 \) by
\[
\begin{align*}
L_1 &= \alpha_1 \left( \frac{\gamma n!}{d(n-1)!} \int_{\theta_0}^T (T - s)^{n-1} c(s)f_\infty^s \, ds \right)^{-1}, \\
L_2 &= \alpha_1 \left( \frac{T^n}{d(n-1)!} \int_0^T (T - s)^{n-1} c(s)f_\infty^s \, ds \right)^{-1}, \\
L_3 &= \alpha_2 \left( \frac{\gamma n!}{e(m-1)!} \int_{\theta_0}^T (T - s)^{m-1} d(s)g_\infty^i \, ds \right)^{-1}, \\
L_4 &= \alpha_2 \left( \frac{T^n}{e(m-1)!} \int_0^T (T - s)^{m-1} d(s)g_\infty^i \, ds \right)^{-1}.
\end{align*}
\]

**Theorem 2.** Assume that (H1), (H2) and (H3) hold and \( \alpha_1, \alpha_2 > 0 \) are positive numbers such that \( \alpha_1 + \alpha_2 = 1 \).

a) If \( f_0^s, g_0^s, f_\infty^i, g_\infty^i \in (0, \infty) \), \( L_1 < L_2 \) and \( L_3 < L_4 \), then for each \( \lambda \in (L_1, L_2) \) and \( \mu \in (L_3, L_4) \) there exists a positive solution \( (u(t), v(t)) \), \ t \in [0,T] \ for \( (S) -(BC) \).

b) If \( f_0^s = g_0^s = 0, f_\infty^i, g_\infty^i \in (0, \infty) \), then for each \( \lambda \in (L_1, \infty) \) and \( \mu \in (L_3, \infty) \) there exists a positive solution \( (u(t), v(t)) \), \ t \in [0,T] \ for \( (S) -(BC) \).

c) If \( f_0^s, g_0^s \in (0, \infty) \), \( f_\infty^i = g_\infty^i = \infty \), then for each \( \lambda \in (0, L_2) \) and \( \mu \in (0, L_4) \) there exists a positive solution \( (u(t), v(t)) \), \ t \in [0,T] \ for \( (S) -(BC) \).

d) If \( f_0^s = g_0^s = 0, f_\infty^i = g_\infty^i = \infty \), then for each \( \lambda \in (0, \infty) \) and \( \mu \in (0, \infty) \) there exists a positive solution \( (u(t), v(t)) \), \ t \in [0,T] \ for \( (S) -(BC) \).

**Sketch of proof.** a) We suppose \( f_0^s, g_0^s, f_\infty^i, g_\infty^i \in (0, \infty) \), \( L_1 < L_2 \) and \( L_3 < L_4 \). Let \( P_1, P_2 : Y \to X \) and \( Q : Y \to Y \) be the operators defined by
\[
\begin{align*}
P_1(u,v)(t) &= \lambda \int_0^T G_1(t,s)c(s)f(u(s), v(s)) \, ds, \ t \in [0,T], \\
P_2(u,v)(t) &= \mu \int_0^T G_2(t,s)d(s)g(u(s), v(s)) \, ds, \ t \in [0,T],
\end{align*}
\]
and \( Q(u, v) = (P_1(u, v), P_2(u, v)) \), \((u, v) \in Y\), where \( G_1, G_2 \) are the Green’s functions defined in Section 2.

The solutions of problem (S) \(- (BC)\) are the fixed points of the operator \( Q \).

We consider an arbitrary element \((u, v) \in C\). Because \( P_1(u, v) \) and \( P_2(u, v) \) satisfy the problem (1)-(2) for \( y(t) = \lambda c(t)f(u(t), v(t)), \) \( t \in [0, T] \), and the problem (3)-(4) for \( h(t) = \mu d(t)g(u(t), v(t)), \) \( t \in [0, T] \), respectively, then by Lemma 5, we obtain

\[
\inf_{t \in [\theta_0, T]} P_1(u, v)(t) = \gamma_1 \|P_1(u, v)\|, \quad \inf_{t \in [\theta_0, T]} P_2(u, v)(t) = \gamma_2 \|P_2(u, v)\|.
\]

Therefore we deduce

\[
\inf_{t \in [\theta_0, T]} [P_1(u, v)(t) + P_2(u, v)(t)] = \gamma_1 \|P_1(u, v)\| + \gamma_2 \|P_2(u, v)\| = \gamma \|Q(u, v)\|_Y.
\]

By using Lemma 3, (H2) and (H3), we obtain that \( P_1(u, v)(t) \geq 0, P_2(u, v)(t) \geq 0 \), for all \( t \in [0, T] \), and so we deduce that \( Q(u, v) \in C \). Hence we get \( Q(C) \subset C \).

By using standard arguments, we can easily show that \( P_1 \) and \( P_2 \) are completely continuous, and then \( Q \) is a completely continuous operator.

Now let \( \lambda \in (L_1, L_2), \mu \in (L_3, L_4) \), and let \( \varepsilon > 0 \) be a positive number such that \( \varepsilon < f_i^*, \varepsilon < g_i^* \) and

\[
\begin{align*}
\alpha_1 \left( \frac{\gamma c_{p-2}^n}{d(n-1)!} \int_0^T (T-s)^{n-1} c(s)(f_i^* - \varepsilon) \, ds \right)^{-1} & \leq \lambda, \\
\alpha_1 \left( \frac{T^{n-1}}{d(n-1)!} \int_0^T (T-s)^{n-1} c(s)(f_0^* + \varepsilon) \, ds \right)^{-1} & \geq \lambda, \\
\alpha_2 \left( \frac{\gamma c_{p-2}^m}{e(m-1)!} \int_0^T (T-s)^{m-1} g(s)(g_i^* - \varepsilon) \, ds \right)^{-1} & \leq \mu, \\
\alpha_2 \left( \frac{T^{m-1}}{e(m-1)!} \int_0^T (T-s)^{m-1} g(s)(g_0^* + \varepsilon) \, ds \right)^{-1} & \geq \mu.
\end{align*}
\]

By (H3), we deduce that there exists \( K_1 > 0 \) such that for all \( u, v \in \mathbb{R}_+ \), with \( 0 \leq u + v \leq K_1 \), we have \( f(u, v) \leq (f_i^* + \varepsilon)(u + v) \) and \( g(u, v) \leq (g_i^* + \varepsilon)(u + v) \).

We define the ball \( \Omega_1 = \{(u, v) \in Y, \|y\|_Y < K_1\} \). Now let \((u, v) \in C \cap \partial \Omega_1\), that is \((u, v) \in C\) with \( \|y\|_Y = K_1 \) or, equivalently, \( \|u\| + \|v\| = K_1 \). Then \( u(t) + v(t) \leq K_1 \) for all \( t \in [0, T] \). By Lemma 4, after some computations, we deduce that \( P_1(u, v)(t) \leq \alpha_1 \|y\|_Y \) and \( P_2(u, v)(t) \leq \alpha_1 \|y\|_Y \). Therefore \( \|y\|_Y \leq \alpha_1 \|y\|_Y \). In a similar manner, we obtain \( \|y\|_Y \leq \alpha_1 \|y\|_Y \).

Then for \((u, v) \in C \cap \partial \Omega_1\) we deduce

\[
\|Q(u, v)\|_Y = \|(P_1(u, v), P_2(u, v))\|_Y \leq \alpha_1 \|y\|_Y + \alpha_2 \|y\|_Y = \|y\|_Y.
\]

By the definitions of \( f_{i\infty}^* \) and \( g_{i\infty}^* \), there exists \( K_2 > 0 \) such that \( f(u, v) \geq (f_{i\infty}^* - \varepsilon)(u + v) \) and \( g(u, v) \geq (g_{i\infty}^* - \varepsilon)(u + v) \) for all \( u, v \geq 0 \), with \( u + v \geq K_2 \). We consider \( K_2 = \)
max\{2K_1, \bar{K}_2/r\}, and we define \(\Omega_2 = \{(u, v) \in Y, \| (u, v) \|_Y < K_2\}\). Then for \((u, v) \in C\) with \(\| (u, v) \|_Y = K_2\), we obtain

\[
u(t) + \mu(t) \geq \gamma_1 \| u \| + \gamma_2 \| v \| \geq \gamma (\| u \| + \| v \|) = \gamma \| (u, v) \|_Y = \gamma K_2 \geq \bar{K}_2, \quad \forall t \in [\theta_0, T].
\]

Then by Lemma 4, after some computations, we deduce that \(P_1(u, v)(\xi_{p-2}) \geq \alpha_1 \| (u, v) \|_Y\). So \(\| P_1(u, v) \| \geq P_1(u, v)(\xi_{p-2}) \geq \alpha_1 \| (u, v) \|_Y\). In a similar manner, we obtain \(\| P_2(u, v) \| \geq P_2(u, v)(\eta_{q-2}) \geq \alpha_2 \| (u, v) \|_Y\).

Hence for \((u, v) \in C \cap \partial \Omega_2\) we obtain

\[
\| P_1(u, v) \|_Y = \| P_2(u, v) \|_Y \geq (\alpha_1 + \alpha_2) \| (u, v) \|_Y = \| (u, v) \|_Y.
\]

By using Theorem 1 i) with \(T = Q, K = C, a = K_1, b = K_2, K(a, b) = C \cap (\Omega_2 \setminus \Omega_1), K_a = C \cap \partial \Omega_1, K_b = C \cap \partial \Omega_2\), we deduce that \(Q\) has a fixed point \((u, v) \in C \cap (\Omega_2 \setminus \Omega_1)\) such that \(K_1 \leq \| (u, v) \|_Y \leq K_2\) or \(K_1 \leq \| u \| + \| v \| \leq K_2\).

The proofs of cases b)-d) are similar to that of case a) and we shall omit them (see also the paper [1]).

\[\Box\]

**Remark 1.** The condition \(L_1 < L_2\) from Theorem 2 is equivalent to

\[
f_0^T (T - s)^{n-1} c(s) \, ds < f_0^T \gamma c_{p-2}^{n-1} \int_0^T (T - s)^{n-1} c(s) \, ds,
\]

and \(L_3 < L_4\) is equivalent to

\[
g_0^T (T - s)^{m-1} \, ds < g_0^T \gamma g_{q-2}^{m-1} \int_0^T (T - s)^{m-1} \, ds.
\]

In what follows, for \(f_0^T, f_0^T, f_\infty^T, g_\infty^T \in (0, \infty)\) and positive numbers \(\alpha_1, \alpha_2 > 0\) such that \(\alpha_1 + \alpha_2 = 1\), we define the positive numbers \(\bar{L}_1, \bar{L}_2, \bar{L}_3\) and \(\bar{L}_4\) by

\[
\bar{L}_1 = \alpha_1 \left( \frac{\gamma c_{p-2}^{n-1}}{d(n-1)!} \int_0^T (T - s)^{n-1} c(s) f_0^T \, ds \right)^{-1},
\]

\[
\bar{L}_2 = \alpha_1 \left( \frac{T^{n-1}}{d(n-1)!} \int_0^T (T - s)^{n-1} c(s) f_\infty^T \, ds \right)^{-1},
\]

\[
\bar{L}_3 = \alpha_2 \left( \frac{\gamma g_{q-2}^{m-1}}{e(m-1)!} \int_0^T (T - s)^{m-1} g_0^T \, ds \right)^{-1},
\]

\[
\bar{L}_4 = \alpha_2 \left( \frac{T^{m-1}}{e(m-1)!} \int_0^T (T - s)^{m-1} g_\infty^T \, ds \right)^{-1}.
\]

**Theorem 3.** Assume that (H1), (H2’) and (H3) hold and \(\alpha_1, \alpha_2 > 0\) are positive numbers such that \(\alpha_1 + \alpha_2 = 1\).

a) If \(f_0^T, g_0^T, f_\infty^T, g_\infty^T \in (0, \infty)\), \(\bar{L}_1 \leq \bar{L}_2\) and \(\bar{L}_3 \leq \bar{L}_4\), then for each \(\lambda \in (\bar{L}_1, \bar{L}_2)\) and \(\mu \in (\bar{L}_3, \bar{L}_4)\) there exists a positive solution \((u(t), v(t))\), \(t \in [0, T]\) for \((S) - (BC)\).

b) If \(f_\infty^T = g_\infty^T = 0\), \(f_0^T, g_0^T \in (0, \infty)\), then for each \(\lambda \in (\bar{L}_1, \infty)\) and \(\mu \in (\bar{L}_3, \infty)\) there exists a positive solution \((u(t), v(t))\), \(t \in [0, T]\) for \((S) - (BC)\).
c) If \( f^*_i, g^*_i \in (0, \infty), f^*_0 = g^*_0 = \infty \), then for each \( \lambda \in (0, \bar{L}_2) \) and \( \mu \in (0, \bar{L}_4) \) there exists a positive solution \((u(t), v(t))\), \( t \in [0, T] \) for \((S) - (BC)\).

d) If \( f^*_\infty = g^*_\infty = 0, f^*_0 = g^*_0 = \infty \), then for each \( \lambda \in (0, \infty) \) and \( \mu \in (0, \infty) \) there exists a positive solution \((u(t), v(t)), t \in [0, T]\) for \((S) - (BC)\).

**Sketch of proof.** a) Let \( \lambda \in (\bar{L}_1, \bar{L}_2) \) and \( \mu \in (\bar{L}_3, \bar{L}_4) \). We select a positive number \( \varepsilon \) such that \( \varepsilon < f^*_0, \varepsilon < g^*_0 \) and

\[
\alpha_1 \left( \frac{\gamma s^{-1}}{d(n-1)!} \int_{s^2}^{T} (T-s)^{n-1} c(s)(f^*_0 - \varepsilon) \, ds \right)^{-1} \leq \lambda,
\]

\[
\alpha_1 \left( \frac{T^{n-1}}{d(n-1)!} \int_{0}^{T} (T-s)^{n-1} c(s)(f^*_\infty + \varepsilon) \, ds \right)^{-1} \geq \lambda,
\]

\[
\alpha_2 \left( \frac{\gamma s^{-1}}{e(m-1)!} \int_{s^{-2}}^{T} (T-s)^{m-1} d(s)(g^*_0 - \varepsilon) \, ds \right)^{-1} \leq \mu,
\]

\[
\alpha_2 \left( \frac{T^{m-1}}{e(m-1)!} \int_{0}^{T} (T-s)^{m-1} d(s)(g^*_\infty + \varepsilon) \, ds \right)^{-1} \geq \mu.
\]

We also consider the operators defined in the proof of Theorem 2. By the definitions of \( f^*_0, g^*_0 \in (0, \infty) \), we deduce that there exists \( K_3 > 0 \) such that \( f(u, v) \geq (f^*_0 - \varepsilon)(u + v) \), \( g(u, v) \geq (g^*_0 - \varepsilon)(u + v) \) for all \( u, v \geq 0 \), with \( 0 \leq u + v \leq K_3 \).

We denote by \( \Omega_3 = \{(u, v) \in Y; \|u\| + \|v\| < K_3\} \). Let \((u, v) \in C \) with \( \|(u, v)\| \in Y = \Omega_3 \). By using Lemma 4, we obtain after some computations \( P_1(u, v) = (\xi_{p-2}) \geq \alpha_1 \|u\| Y \). Therefore, \( P_1(u, v) \geq (P_1(u, v))(\xi_{q-2}) \geq \alpha_1 \|u\| Y \). In a similar manner, we obtain \( P_2(u, v) \geq (P_2(u, v))(\eta_{q-2}) \geq \alpha_2 \|u\| Y \).

Thus for an arbitrary element \((u, v) \in C \cap \partial \Omega_3 \) we obtain

\[
\|Q(u, v)\| Y \geq (\alpha_1 + \alpha_2) \|u\| Y = \|(u, v)\| Y.
\]

Now we define the functions \( f^*, g^* : \mathbb{R}_+ \to \mathbb{R}_+ \), \( f^*(x) = \max_{0 \leq u + v \leq x} f(u, v), g^*(x) = \max_{0 \leq u + v \leq x} g(u, v), x \in \mathbb{R}_+ \). Then \( f(u, v) \leq f^*(x), g(u, v) \leq g^*(x) \) for all \((u, v), u \geq 0, v \geq 0 \) and \( 0 \leq u + v \leq x \). The functions \( f^*, g^* \) are nondecreasing and they satisfy the conditions

\[
\limsup_{x \to \infty} \frac{f^*(x)}{x} \leq f^*_\infty, \quad \limsup_{x \to \infty} \frac{g^*(x)}{x} \leq g^*_\infty.
\]

Therefore, for \( \varepsilon > 0 \) there exists \( K_4 > 0 \), such that for all \( x \geq K_4 \), we have

\[
\frac{f^*(x)}{x} \leq \limsup_{x \to \infty} \frac{f^*(x)}{x} + \varepsilon \leq f^*_\infty + \varepsilon, \quad \frac{g^*(x)}{x} \leq \limsup_{x \to \infty} \frac{g^*(x)}{x} + \varepsilon \leq g^*_\infty + \varepsilon,
\]

and so \( f^*(x) \leq (f^*_\infty + \varepsilon)x \) and \( g^*(x) \leq (g^*_\infty + \varepsilon)x \).

We now consider \( K_4 = \max\{2K_3, K_4\} \), and we denote by \( \Omega_4 = \{(u, v) \in Y, \|u\| Y \leq K_4\} \). Let \((u, v) \in C \cap \partial \Omega_4 \). By definitions of \( f^* \) and \( g^* \) we have

\[
f(u(t), v(t)) \leq f^*(\|u(t), v(t)\|), \quad g(u(t), v(t)) \leq g^*(\|u(t), v(t)\|), \quad \forall t \in [0, T].
\]
Then for all \( t \in [0, T] \), after some computations, we obtain \( P_1(u, v)(t) \leq \alpha_1 \|(u, v)\|_Y \), and so \( \|P_1(u, v)\| \leq \alpha_1 \|(u, v)\|_Y \). In a similar manner, we obtain \( \|P_2(u, v)\| \leq \alpha_2 \|(u, v)\|_Y \).

Therefore for \((u, v) \in C \cap \partial \Omega_4\) it follows that
\[
\|Q(u, v)\|_Y \leq (\alpha_1 + \alpha_2) \|(u, v)\|_Y = \|(u, v)\|_Y.
\]

By using Theorem 1 ii) with \( T = Q, K = C, a = K_3, b = K_4, K(a, b) = C \cap (\Omega_4 \setminus \Omega_3) \), \( K_a = C \cap \partial \Omega_3, K_b = C \cap \partial \Omega_4 \), we deduce that \( Q \) has a fixed point \((u, v) \in C \cap (\Omega_4 \setminus \Omega_3)\) such that \( K_3 \leq \|(u, v)\|_Y \leq K_4 \).

The proofs of cases b)-d) are similar to that of case a) and we shall omit them (see also the paper [1]).

\[ \Box \]

**Remark 2.** The condition \( \bar{L}_1 < \bar{L}_2 \) is equivalent to
\[
f_s^a T^{m-1} \int_0^T (T - s)^{m-1} c(s) \, ds \leq f_0^\gamma \xi_{p-2}^{-1} \int_{\xi_{p-2}}^T (T - s)^{n-1} c(s) \, ds
\]
and the condition \( \bar{L}_3 < \bar{L}_4 \) is equivalent to
\[
g_s^a T^{m-1} \int_0^T (T - s)^{m-1} d(s) \, ds \leq g_0^\gamma \eta_{q-2}^{-1} \int_{\eta_{q-2}}^T (T - s)^{n-1} d(s) \, ds
\]

**4 Examples**

Let \( T = 1, n = 3, m = 4, p = 5, q = 4, c(t) = c_0 t, d(t) = d_0 t, \) for \( t \in [0, 1] \), with \( c_0, d_0 > 0 \), \( \xi_1 = \frac{1}{4}, \xi_2 = \frac{1}{2}, \xi_3 = \frac{3}{4}, \eta_1 = \frac{1}{3}, \eta_2 = \frac{2}{3}, \alpha_1 = 1, \alpha_2 = \frac{1}{2}, \alpha_3 = \frac{1}{4}, b_1 = 1, b_2 = 2 \).

We have \( d = \frac{5}{8}, e = \frac{10}{27}, \theta_0 = \frac{3}{4}, \gamma_1 = \frac{1}{16}, \gamma_2 = \frac{1}{27}, \gamma = \frac{1}{27}. \)

We consider the higher-order differential system
\[
(S_1) \quad \begin{cases}
  w^{(3)}(t) + \lambda c_0 tf(u(t), v(t)) = 0, & t \in (0, 1), \\
  v^{(4)}(t) + \mu d_0 tg(u(t), v(t)) = 0, & t \in (0, 1),
\end{cases}
\]
with the boundary conditions
\[
(BC_1) \quad \begin{cases}
  u(0) = u'(0) = 0, & u(1) = u(\frac{1}{4}) + \frac{1}{2} u(\frac{1}{2}) + \frac{1}{4} u(\frac{3}{4}), \\
  v(0) = v'(0) = v''(0) = 0, & v(1) = v(\frac{1}{4}) + 2 v(\frac{3}{4}).
\end{cases}
\]

1. First we consider the functions
\[
f(u, v) = \frac{(u + v)(p_1 u + 1)(q_1 + \sin v)}{u + 1}, \quad g(u, v) = \frac{(u + v)(p_2 v + 1)(q_2 + \cos u)}{v + 1},
\]
with \( p_1, p_2 > 0, q_1, q_2 > 1 \).

It follows that \( f_0^\gamma = f_0^\gamma = q_1, g_0^\gamma = g_0^\gamma = q_2 + 1, f_\infty = p_1(q_1 + 1), f_\infty = p_1(q_1 - 1), g_\infty = p_2(q_2 + 1), g_\infty = p_2(q_2 - 1). \)

The constants \( L_i, i = 1, 4 \) from Section 3 are of the form
\[
L_1 = \frac{184320 \alpha_1}{13 c_0 p_1 (q_1 - 1)}, \quad L_2 = \frac{15 \alpha_1}{c_0 q_1}, \quad L_3 = \frac{259200 \alpha_2}{d_0 p_2 (q_2 - 1)}, \quad L_4 = \frac{400 \alpha_2}{9 d_0 (q_2 + 1)}.
\]
and the conditions $L_1 < L_2$ and $L_3 < L_4$ are equivalent to

$$\frac{q_1}{p_1(q_1 - 1)} < \frac{13}{12288}, \quad \frac{q_2 + 1}{p_2(q_2 - 1)} < \frac{1}{5832}. $$

We apply Theorem 2 a) for $\alpha_1, \alpha_2 > 0$ with $\alpha_1 + \alpha_2 = 1$. If the above conditions are satisfied, then for each $\lambda \in (L_1, L_2)$ and $\mu \in (L_3, L_4)$, there exists a positive solution $(u(t), v(t))$, $t \in [0, T]$ for problem $(S_1) - (BC_1)$.

2. We consider the functions

$$ f(u, v) = (u + v)^{\beta_1}, \quad g(u, v) = (u + v)^{\beta_2}, \quad u, v \in [0, \infty), $$

with $\beta_1, \beta_2 > 1$. Then $f_0^s = f_0^i = g_0^s = g_0^i = 0$ and $f_\infty^s = f_\infty^i = g_\infty^s = g_\infty^i = \infty$. By Theorem 2 d) we deduce that for each $\lambda \in (0, \infty)$ and $\mu \in (0, \infty)$ there exists a positive solution $(u(t), v(t))$, $t \in [0, T]$ for problem $(S_1) - (BC_1)$.

3. We consider the functions

$$ f(u, v) = (u + v)^{\gamma_1}, \quad g(u, v) = (u + v)^{\gamma_2}, \quad u, v \in [0, \infty), $$

with $\gamma_1, \gamma_2 \in (0, 1)$. Then $f_0^s = f_0^i = g_0^s = g_0^i = \infty$ and $f_\infty^s = f_\infty^i = g_\infty^s = g_\infty^i = 0$. By Theorem 3 d) we deduce that for each $\lambda \in (0, \infty)$ and $\mu \in (0, \infty)$ there exists a positive solution $(u(t), v(t))$, $t \in [0, T]$ for problem $(S_1) - (BC_1)$.

**Acknowledgement.** The work of the second author was supported by a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0557.

**References**


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