A GENERALIZATION OF STRICTLY CONVERGENT POWER SERIES AND APPLICATIONS

In memory of Professor Nicolae Popescu

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Abstract

A representation of strictly convergent power series as Newton interpolating series is given. In the case of one indeterminate bounded Newton interpolating series are studied as a generalization of strictly convergent power series. A method for analytic p-adic continuation by means of bounded Newton interpolating series is presented.

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1 Introduction

Let $R$ be a commutative ring with identity and $S = \{(\alpha_{k,1}, ..., \alpha_{k,n})\}_{k \geq 1}$ a fixed sequence of elements of $R^n$. In Section 2 we define the $R$-algebra of Newton interpolating series in $n$ variables denoted by $R_S[[X]]$. Algebraic properties of $K_S[[X]]$, when $K$ is a local field are presented in [5].

If $R$ is a commutative ring with identity and $\|\|$ is a non-trivial non-archimedean norm on $R$ with $\|1\| = 1$, then $(R, \|\|)$ is called a normed ring. We consider the sets (see [1], Chapter 1): $\overset{\circ}{R} = \{x \in R : \|x\| \leq 1\}$, $\overset{\circ}{V} = \{x \in R : \|x\| < 1\}$. Then $\overset{\circ}{R}$ is a commutative ring with identity and $\overset{\circ}{V}$ is an ideal in $\overset{\circ}{R}$. We denote the residue ring $\overset{\circ}{R} / \overset{\circ}{V}$ by $\overset{\sim}{R}$. If $R$ is an integral domain with a non-trivial non-archimedean multiplicative norm, hence an absolute value $\|\|$, then $(R, \|\|)$ is called a valued ring. If $(K, \|\|)$ is a valued field and $(R, \|\|)$ is a valued ring which is a $K$-algebra we suppose that the absolute value of $R$ extends that of $K$.

Let $R$ be a complete non-archimedean normed ring and $R < X >$ the $R$-algebra of

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strictly convergent (restricted formal) power series (see [1], p.35, [4] or [8]). Useful generalizations are given in [6] (so-called separated power series) and [2] (strictly analytic functions defined on a class of domains called analoid sets). If \( R = \mathbb{C}_p \) endowed with the \( p \)-adic absolute value, it is known when a Mahler series may be represented as a strictly convergent power series (see [8], p.354). In Section 3, by means of an arbitrary sequence \( S \) of elements of \( R \), in the case of \( n \) variables, we define \( \mathcal{H}R_S[[X]] \) an \( R \)-subalgebra of \( R_S[[X]] \) which is a Banach algebra with respect to the Gauss norm. Theorem 2 from Section 3 shows that the algebra of strictly convergent power series \( R < X > \) and \( \mathcal{H}R_S[[X]] \) are isometrically isomorphic.

In Section 4, \( K \) is a complete valued field having its residue field at most countable and \( T \) is a fixed set of representatives of the residue field in the valuation ring. By means of \( T \) we construct a sequence \( S_T \) such that every element of \( T \) appears infinitely many times in \( S_T \). In the case \( n = 1 \), we study the \( K \)-subalgebra \( BK_{S_T}[[X]] \) of \( K_{S_T}[[X]] \) which contains the series having bounded coefficients. By Theorem 2 these series are generalization of strictly convergent power series. With respect to Gauss norm \( BK_{S_T}[[X]] \) is a Banach algebra such that \( BK[[X]] \), the \( K \)-algebra of formal power series with bounded coefficients, is homeomorphic to a residue algebra of \( BK_{S_T}[[X]] \) by a closed ideal (see Theorem 4). Moreover for every \( f \in BK[[X]] \) there exists a series of \( g \in BK_{S_T}[[X]] \) such that the corresponding functions defined on the maximal ideal of the valuation ring are equal (see Corollary 2). Theorem 5 with its corollary deal with properties of zeros of associated functions to the elements of \( BK_{S_T}[[X]] \). Theorem 6 is Identity Theorem for the elements of \( BK_{S_T}[[X]] \).

It is well known that the analytic continuation in the \( p \)-adic analysis cannot be achieved by means of Taylor expansions. By means of Krasner’s method it is possible to define analytic elements on the unit open ball for a set of functions defined by bounded power series which satisfy Christol-Robba’s condition but there are simple examples of functions which do not belong to this set. If \( K = \mathbb{C}_p \), we define in Section 5 so-called Newton analytic elements which extend on the unit ball the usual analytic elements (see [3] or [8]). In this manner we define analytic continuation of bounded power series even in the case when the conditions of Christol-Robba’s Theorem do not hold (see Remark 1).

## 2 Basic notations and definitions

Let \( n \) be a fixed positive integer. If \( \nu = (\nu_1, \nu_2, ..., \nu_n) \in \mathbb{N}^n \), we set \( N(\nu) = \nu_1 + \nu_2 + ... + \nu_n \), for every \( i = 1, 2, ..., n \), and \( 0 = (0, ..., 0) \in \mathbb{N}^n \). For \( \nu, \tau \in (\tau_1, \tau_2, ..., \tau_n) \in \mathbb{N}^n \), \( j \in \mathbb{N} \), we define \( \nu + \tau = (\nu_1 + \tau_1, ..., \nu_n + \tau_n) \) and \( j\nu = (j\nu_1, j\nu_2, ..., j\nu_n) \). We set \( \nu < l \) if \( \nu \) is less than \( \tau \) with respect to the following lexicographical order: \( \nu_s < \tau_s \), where \( s \) is the greatest positive integer less than \( n \) such that \( \nu_s = \tau_s \). We order also \( \mathbb{N}^n \) in the following way: \( \nu <_o \tau \) if either \( N(\nu) < N(\tau) \) or \( N(\nu) = N(\tau) \) and \( \nu < l \). We denote by \( \infty^n \) a symbol such that \( \nu <_o \infty^n \) for every \( \nu \in \mathbb{N}^n \). It is obvious that for a fixed \( \tau \in \mathbb{N}^n \), the set \( \{ \nu \in \mathbb{N}^n : \nu \leq_o \tau \} \) is finite.

Let \( R \) be a commutative ring with identity and \( S = \{ (\alpha_{k,1}, ..., \alpha_{k,n}) \}_{k \geq 1} \) a fixed sequence of elements of \( R^n \). In the polynomial ring \( R[X] = R[X_1, ..., X_n] \) we construct by
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Recurrence, with respect to the defined order $<_o$ of $\mathbb{N}^n$, the polynomials

$$U_0 = 1, U_{(1,0,...,0)} = X_1 - \alpha_{1,1},..., U_{(0,0,...,1)} = X_n - \alpha_{1,n}$$

and generally for every $\tau = (\tau_1, \tau_2,...,\tau_n) \in \mathbb{N}^n$,

$$U_\tau = \prod_{0<j \leq \pi_1(\tau)} (X_1 - \alpha_{j,1}) \prod_{0<j \leq \pi_2(\tau)} (X_2 - \alpha_{j,2}) \cdots \prod_{0<j \leq \pi_n(\tau)} (X_n - \alpha_{j,n}), \quad (1)$$

where $\pi_i(\tau) = \tau_i$. If, for each $\tau \in \mathbb{N}^n$, we consider the principal ideal of $R[X] \mathcal{I}_\tau = <U_\tau>$, then $\{\mathcal{I}_\tau\}_{\tau \in \mathbb{N}^n}$ is a system of neighborhoods of zero of the polynomial ring. Thus $R[X]$ becomes a topological Hausdorff space with respect to this topology denoted by $\mathcal{T}_S$. We consider $R_S[[X]]$ the completion of $R[X]$ with respect to $\mathcal{T}_S$. It is easy to prove that we can represent $R_S[[X]]$ as the set of formal series

$$R_S[[X]] = \left\{ f = \sum_{\tau=0}^{\infty} a_\tau U_\tau \mid a_\tau \in R \right\}, \quad (2)$$

where in each series the order of terms are given by $<_o$, two such expressions being regarded as equal if and only if they have the same coefficients. We call an element $f$ from $R_S[[X]]$ a (formal) Newton interpolating series with coefficients in $R$ defined by the sequence $S$. If $n = 1$, $R[X] = R[X]$ and $S = \{\alpha_k\}_{k \geq 1}$, then the polynomials $u_i$ defined by (1) can be written in the form

$$u_0 = 1, \quad u_i = \prod_{j=1}^{i} (X - \alpha_j), \quad i \geq 1. \quad (3)$$

Since, for every nonnegative integer $j$,

$$X^j = u_j + \sum_{i=1}^{j} q_{i,j}(\alpha_1, \ldots, \alpha_{j-i+1}) u_{j-i}, \quad (4)$$

where $q_{i,j}$ are homogeneous polynomials of degree $i$ with integral coefficients (i.e. belonging to the canonical homomorphic image of $\mathbb{Z}$ in $R$), it follows that every polynomial $P = \sum_{i=0}^{p} b_i X^i \in R[X]$ can be written uniquely in the form

$$P = \sum_{i=0}^{p} a_i u_i, \quad (5)$$

where

$$a_i = b_i + \sum_{j=i+1}^{p} b_j Q_{i,j}(\alpha_1, \ldots, \alpha_{i+1}), \quad (6)$$

and $Q_{i,j}$ are homogeneous polynomials with integral coefficients. Hence if $u_i$, $u_j$ are given by (3), we obtain that for every $k$ such that $\max \{i,j\} \leq k \leq i + j$, there exist in $R$ the
elements $d_k(i,j)$ uniquely defined such that
\[ u_iu_j = \sum_{k=\max\{i,j\}}^{i+j} d_k(i,j)u_k. \] (7)

Now we consider $P = \sum_{\nu \leq o^\tau} b_\nu X^\nu \in R[[X]]$. From (5) and (6), by induction on $n$, it follows that
\[ P = \sum_{\nu \leq o^\tau} a_\nu U_\nu, \quad a_\nu \in R. \] (8)

If $f, g = \sum_{\nu=0}^{\infty} b_\nu U_\nu \in R_S[[X]]$, we define addition and multiplication of $f$ and $g$ as follows:
\[ f + g = \sum_{\nu=0}^{\infty} (a_\nu + b_\nu) U_\nu, \] (9)
\[ fg = \sum_{\nu=0}^{\infty} p_\nu U_\nu, \] (10)
where
\[ p_\nu = \sum_{\mu, \theta \in I(\nu)} D_\nu(\mu, \theta) a_\mu b_\theta, \] (11)

$D_\nu(\mu, \theta) = d_{\nu_1}(\mu_1, \theta_1) \ldots d_{\nu_n}(\mu_n, \theta_n)$, $d_i(s, t)$ are defined in (7) and $I(\nu) = \{ (\mu, \theta) \in \mathbb{N}^n \times \mathbb{N}^n : \max\{\mu, \theta\} \leq o^\nu, \mu + \theta \geq o^\nu \}$. Thus with respect to these definitions of addition and multiplication, $R_S[[X]]$ becomes a complete Hausdorff topological commutative $R$-algebra which contains $R[X]$. Moreover by (1), (9)-(11) it follows that as $R$-algebras
\[ R_{S^{n-1}}[X^{(n-1)}]_{S_n}[X_n] \cong R_S[[X]], \] (12)
where $S^{n-1} = \{ (\alpha_{k,1}, \ldots, \alpha_{k,n-1}) \}_{k \geq 1}$, $X^{(n-1)} = (X_1, \ldots, X_{n-1})$ and $S_n = \{ \alpha_{k,n} \}_{k \geq 1}$.

3 A representation of strictly convergent power series

Let $(R, \| \|)$ be a normed ring and $S = \{ (\alpha_{k,1}, \ldots, \alpha_{k,n}) \}_{k \geq 1}$ a fixed sequence of elements of $R_n$. We consider
\[ \mathcal{H}R_S[[X]] = \left\{ f = \sum_{\nu=0}^{\infty} a_\nu U_\nu \in R_S[[X]] : \lim_{N(\nu) \to \infty} \| a_\nu \| = 0 \right\}. \] (13)

If $f = \sum_{\nu=0}^{\infty} a_\nu U_\nu \in \mathcal{H}R_S[[X]]$, then we define
\[ \| f \|_{\mathcal{H}R_S[[X]]} = \sup_{\nu} \| a_\nu \|. \] (14)
Theorem 1. If $R$ is a normed (resp. valued) ring and $S$ is a fixed sequence of elements of $R^n$, then $\mathcal{H}R_S[[X]]$ is a $R$-subalgebra of $R_S[[X]]$ and $|| \ ||$ defined by (14) is a non-archimedean norm (resp. absolute value) on $\mathcal{H}R_S[[X]]$. Moreover if $R$ is a complete normed (resp. valued) ring, then $\mathcal{H}R_S[[X]]$ becomes a Banach $R$-algebra which is the completion of $R[X]$ with respect to the metric defined by the norm (resp. absolute value).

Proof. First suppose $n = 1$. Let $f, g = \sum_{i=0}^{\infty} b_i u_i$ be elements of $\mathcal{H}R_S[[X]]$. Then, by (9) and (14), with $n = 1$, we obtain $\|f \pm g\| = \sup_i \{\|a_i \pm b_i\|\} \leq \max\{\|f\|, \|g\|\}$. Similarly, by (7), (10) and (11), since $u_i \in \overset{\circ}{R} [X]$, it follows that $d_k(i,j) \in \overset{\circ}{R}$ and $\|fg\| = \sup_i \|p_i\| \leq \|f\||g\|$. If $R$ is a valued ring we choose $\psi = \phi$ such that $\|\|\|_i$ is the greatest index with this property. Since $\mathcal{H}R_S[[X]]$ is complete it follows that $\mathcal{H}R_S[[X]]$ resp. valued ring we choose $\phi = \psi$ such that $\|\|_i$ is the greatest index with this property. Since $\mathcal{H}R_S[[X]]$ is complete it follows that $\mathcal{H}R_S[[X]]$ resp. valued $\phi = \psi$ such that $\|\|_i$ is the greatest index with this property. Since $\mathcal{H}R_S[[X]]$ is complete it follows that $\mathcal{H}R_S[[X]]$.

When $R$ is complete it follows that $\mathcal{H}R_S[[X]]$ is complete because it is isometrically isomorphic, as an $R$-module, to $c(R)$, the space of zero sequences over $R$ (see [1], Proposition 6, Sec. 2.1). Now the theorem follows by induction on $n$ by using (12). $\square$

Theorem 2. If $R$ is a complete normed ring and $S = \{\alpha_k\}_{k \geq 1}$ is a fixed sequence of elements of $R$, then the Banach $R$-algebra $\mathcal{H}R_S[[X]]$ is isometrically isomorphic to the $R$-algebra $R < X >$ of strictly convergent power series.

Proof. If $P = \sum_{i=0}^{p} b_i X^i \in R[X]$, then it can be written also in the form (5), where $a_i$ are given in (6). Similarly we obtain

$$b_i = a_i + \sum_{j=i+1} a_j T_{i,j}(\alpha_1, \ldots, \alpha_j),$$

(15)

where $T_{i,j}$ are homogeneous polynomial with integral coefficients. Suppose $\|P\|_{\mathcal{H}R_S[[X]]} = \|a_{i_0}\|$, where $i_0$ is the greatest index with this property. Since $\|T_{i,j}(\alpha_1, \ldots, \alpha_{i+1})\| \leq 1$, it follows that $\|b_{i_0}\| = \|a_{i_0}\|$ and $\|b_i\| \leq \max_{j \geq i} \|a_j\|$. Hence $\|P\|_{R < X >} = \|P\|_{\mathcal{H}R_S[[X]]}$.

Now, by means of (6) we define $\phi : R < X > \to \mathcal{H}R_S[[X]]$ such that

$$\phi \left( \sum_{i=0}^{\infty} b_i X^i \right) = \sum_{i=0}^{\infty} a_i u_i,$$

(16)

where

$$a_i = b_i + \sum_{j=i+1}^{\infty} b_j Q_{i,j}(\alpha_1, \ldots, \alpha_{i+1}).$$

(17)

Similarly, by using (15), we can define $\psi : \mathcal{H}R_S[[X]] \to R < X >$ such that

$$\psi \left( \sum_{i=0}^{\infty} a_i u_i \right) = \sum_{i=0}^{\infty} b_i X^i,$$

(18)
where
\[ b_i = a_i + \sum_{j=i+1}^{\infty} a_j T_{i,j}(\alpha_1, \ldots, \alpha_j). \] (19)

Then the mappings \( \phi \) and \( \psi \) are well defined and continuous with respect to the corresponding norms. By (16)-(19) we obtain that the restricted mappings \( \phi \) and \( \psi \) are inverse to each other on \( \mathcal{R}[X] \). Since \( \mathcal{R}[X] \) is dense in \( \mathcal{H}K_S[[X]] \) it follows that \( \phi \) and \( \psi \) are inverse to each other and hence we obtain that \( \phi \) is bijective. In fact \( \phi \) is the identity map on \( \mathcal{R}[X] \) so \( \phi \) is also a \( \mathcal{R} \)-algebra morphism. So we obtain that \( \mathcal{R}[X] \) and \( \mathcal{H}K_S[[X]] \) are isomorphic \( \mathcal{R} \)-algebras. □

**Corollary 1.** If \( K \) is a complete valued field and \( S = \{(\alpha_{k,1}, \ldots, \alpha_{k,n})\}_{k \geq 1} \) is a fixed sequence of elements of \( \mathcal{K} \), then the algebra of strictly convergent power series \( K < X > \) is isometrically isomorphic to \( \mathcal{H}K_S[[X]] \).

**Proof.** The corollary follows from (12) and Theorem 2. □

4 Bounded Newton interpolating series

In this section \( K \) will denote a complete valued field having its residue field at most countable. For \( a \in K \) and \( r \) a positive real number, we put \( B^+(a, r) = \{x \in K : |x - a| \leq r\} \) and \( B^-(a, r) = \{x \in K : |x - a| < r\} \). We choose \( T = \{\beta_j\}_{j \geq 1} \) a fixed set, at most countable, of elements in \( \mathcal{K} \) and we construct a sequence \( S_T = \{\alpha_i\}_{i \geq 1} \) of elements of \( T \).

By using (3) we define the \( K \)-algebra \( K_{S_T}[[X]] \) with
\[ u_i = \prod_{j=1}^{i} (X - \alpha_j) = \prod_{j=1}^{m(i)} (X - \beta_j)^{\theta(i,j)}, \] (20)
where \( m(i) \) is the number of distinct \( X - \beta_j \) which divides \( u_i(X) \). We consider
\[ BK_{S_T}[[X]] = \left\{ f = \sum_{i=0}^{\infty} a_i u_i \in K_{S_T}[[X]] : \exists M > 0, |a_i| < M, \forall i \right\}. \] (21)

We call an element \( f \) from \( BK_{S_T}[[X]] \) a *bounded Newton interpolating series* with coefficients in \( K \) defined by the sequence \( S_T \). If \( f = \sum_{i=0}^{\infty} a_i u_i \in BK_{S_T}[[X]] \), the real number
\[ \|f\|_{BK_{S_T}[[X]]} = \sup_i |a_i| \] (22)
is well defined. As usual we call \( \|f\|_{BK_{S_T}[[X]]} \), given in (22), the *Gauss norm* on \( BK_{S_T}[[X]] \). In the case when \( T = \{\beta_1\} \), \( BK_{S_T}[[X]] \) becomes
\[ BK[[X - \beta_1]] \]
If many times in sequence of elements of $T$

Lemma 1. If convergence of the series $f\in D$ two results, one on continuity and other on Identity Theorem. If $\{d_i\}_{i=0}^{\infty}$ and (7) it follows that $d_i(s,t)\in \tilde{K}$ and (10), (11), (22) imply

$$\|fg\|_{BK_{ST}[[X]]} \leq \max_i \left\{ \max_{(j,k)\in I(i)} |a_i b_k| \right\} \leq \|f\|_{BK_{ST}[[X]]} \|g\|_{BK_{ST}[[X]]}.$$  

(24)

Thus $BK_{ST}[[X]]$ is a subalgebra of $KS_T[[X]]$ and the Gauss norm is a $K$-algebra non-archimedean norm on $BK_{ST}[[X]]$ making it into a Banach $K$-algebra.

Proof. Let $f, g = \sum_{i=0}^{\infty} b_i u_i \in BK_{ST}[[X]]$. By (9) and (22) we obtain $\|f \pm g\|_{BK_{ST}[[X]]} = \sup_i |a_i \pm b_i| \leq \max_i \left\{ \|f\|_{BK_{ST}[[X]]}, \|g\|_{BK_{ST}[[X]]} \right\}.$ Similarly, since $u_i(X) \in \tilde{K}$, by (6) and (7) it follows that $d_i(s,t)\in \tilde{K}$ and (10), (11), (22) imply

$$\|fg\|_{BK_{ST}[[X]]} \leq \max_i \left\{ \max_{(j,k)\in I(i)} |a_i b_k| \right\} \leq \|f\|_{BK_{ST}[[X]]} \|g\|_{BK_{ST}[[X]]}.$$  

(24)

Thus $BK_{ST}[[X]]$ is a subalgebra of $KS_T[[X]]$ and the Gauss norm is a $K$-algebra norm on $BK_{ST}[[X]]$, $BK_{ST}[[X]]$ is complete because it is isometrically isomorphic as $K$-vector space to $b(K)$, the space of bounded sequences over $K$ (see [1], Proposition 6, Sec. 2.1).

Now we choose $T = \{\beta_j\}_{j\geq 1}$ a fixed set of representatives of $\tilde{K}$ in $\tilde{K}$ and $ST = \{\alpha_i\}_{i\geq 1}$ a sequence of elements of $T$ such that every element of $T$ appears infinitely many times in $ST$. Similarly with the case of Tate algebra (see [1], Sec. 5.1) we prove for $BK_{ST}[[X]]$ two results, one on continuity and other on Identity Theorem. If $D \subset K$ is the domain of convergence of the series $f \in BK_{ST}[[X]]$, then obviously $T \subset D$. We have the following

Lemma 1. If $T = \{\beta_j\}_{j\geq 1}$ is a fixed set of representatives of $\tilde{K}$ in $\tilde{K}$, $ST = \{\alpha_i\}_{i\geq 1}$ is a sequence of elements of $T$ such that every element of $T$ appears infinitely many times in $ST$ and $f = \sum_{i=0}^{\infty} a_i u_i \in BK_{ST}[[X]]$, then

a) $\tilde{K} \subset D$;

b) if $f$ converges at $\tilde{x} \in K$, then it converges for every $x \in K$ such that $|x| \leq |\tilde{x}|$;

c) if $x \in \tilde{K}$, then $|f(x)| \leq \|f\|_{BK_{ST}[[X]]}$.

Proof. a) If $x \in \tilde{K}$, then there is a $\beta_j \in T$ such that $|x - \beta_j| < 1$ and for every $i \neq j$, $|x - \beta_i| = 1$. Since $\beta_j$ appears infinitely many times in $ST$, by (20), $\lim_{i \to \infty} \theta(i,j) = \infty$ which implies $\lim_{i \to \infty} a_i u_i(x) = 0$ and $f$ converges at $x$.

b) It is enough to consider $|\tilde{x}| > 1$. Then for every $i$, $|\tilde{x} - \beta_i| = |\tilde{x}|$, $|a_i u_i(x)| \leq |a_i|\max\{1, |x|\} \leq |a_i|\max\{1, |x|\} \leq |a_i u_i(x)|$ and this implies b).

c) If $x \in \tilde{K}$, then $|f(x)| \leq \sup_i |a_i u_i(x)| \leq \sup_i |a_i| = \|f\|_{BK_{ST}[[X]]}$. □

Proposition 1. If $T = \{\beta_j\}_{j\geq 1}$ is a fixed set of representatives of $\tilde{K}$ in $\tilde{K}$ and $ST = \{\alpha_i\}_{i\geq 1}$ is a sequence of elements of $T$ such that every element of $T$ appears infinitely many times in $ST$, then every $f = \sum_{i=0}^{\infty} a_i u_i \in BK_{ST}[[X]]$ defines a continuous function
on $D$, denoted also by $f$, such that $y \to f(y) = \sum_{i=0}^{\infty} a_i u_i(y) \in K$. Moreover, if $x_0 \in D$, then there exists $\beta_j \in T$ such that the series $\sum_{i=0}^{\infty} a_i u_i(x) \converges uniformly to f(x)$ on $B^+ (\beta_j, |x_0 - \beta_j|)$.

Proof. We may suppose $f \neq 0$. If $y \in \overset{\circ}{K}$, then $\lim_{i \to \infty} a_i u_i(y) = 0$ and the series $\sum_{i=0}^{\infty} a_i u_i(y)$ converges to some element of $K$.

If $y_0 \in \overset{\circ}{K}$ we consider a real number $\varepsilon > 0$. By putting $\delta = \frac{\varepsilon}{\| f \|}$ we take $y \in \overset{\circ}{K}$ such that $|y - y_0| < \delta$. Hence it follows that

$$|f(y) - f(y_0)| \leq \sup_i |a_i| |u_i(y) - u_i(y_0)| \leq \| f \| \sup_i |u_i(y) - u_i(y_0)|.$$

Since $u_i(y) - u_i(y_0) = (y - y_0) w_i(y, y_0)$, where $w_i(y, y_0) \in \overset{\circ}{K}$, we obtain that $|f(y) - f(y_0)| < \varepsilon$ and $f$ gives rise to a continuous function on $\overset{\circ}{K}$.

Now, we suppose $y_0 \in D$, $|y_0| > 1$ and we choose a real number $\varepsilon > 0$. We take $y \in D$ such that $|y - y_0| < 1$. Hence it follows that $|a_i u_i(y)| = |a_i y^j| = |a_i y_0^j| = |a_i u_i(y_0)|$. Thus we can choose $i_0$ such that for every $y \in B^-(y_0, 1)$ $|f(y) - S_{i_0}(y)| < \varepsilon$, where $S_i$ is the partial sum of the series $f$. Since $S_{i_0}(y)$ is a continuous function there is $\delta < 1$ such that for every $y \in B^-(y_0, \delta)$, $|S_{i_0}(y) - S_{i_0}(y_0)| < \varepsilon$. Then

$$|f(y) - f(y_0)| \leq \max \{|f(y) - S_{i_0}(y)|, |S_{i_0}(y) - S_{i_0}(y_0)|, |S_{i_0}(y_0) - f(y_0)|\} < \varepsilon$$

and $f$ gives rise to a continuous function on $D$.

Suppose $x_0 \in D$. If $x_0 \in \overset{\circ}{K}$, we choose $\beta_j \in T$ such that $|x_0 - \beta_j| < 1$. Then for every $x \in B^+ (\beta_j, |x_0 - \beta_j|)$ and $k \neq j$, $|x - \beta_k| = 1$. Hence $|a_i u_i(x)| \leq |a_i u_i(x_0)|$ and the series converges uniformly on $B^+(\beta_j, |x_0 - \beta_j|)$.

If $|x_0| > 1$, then for every $\beta_j \in T$, $|x_0 - \beta_j| = |x_0|$. Thus for every $x \in B^+(\beta_j, |x_0 - \beta_j|) = B^+(0, |x_0|)$, $|a_i u_i(x)| \leq |a_i u_i(x_0)|$, which implies the proposition. $\square$

Theorem 4. Let $T = \{\beta_j\}_{i \geq 1}$ be a fixed set of representatives of $\overset{\circ}{K}$ in $\overset{\circ}{K}$ and let $S_T = \{\alpha_k\}_{k \geq 1}$ be a sequence of elements of $T$. If there exists $\beta_k \in T$ which appears infinitely many times in $S_T$, then there exists a $K$-algebra homomorphism $\varphi : BK_{S_T}[\{X\}] \to BK[[X - \beta_k]]$ such that:

a) $\varphi$ is a continuous $K$-algebra homomorphism from $BK_{S_T}[\{X\}]$ onto $BK[[X - \beta_k]]$;

b) for every $g \in BK_{S_T}[\{X\}]$ and $x \in B^-(\beta_k, 1)$, $g(x) = \varphi(g)(x)$;

c) the induced isomorphism $\bar{\varphi} : BK_{S_T}[\{X\}]/\text{Ker} \varphi \to BK[[X]]$ is a homeomorphism, where $BK_{S_T}[\{X\}]/\text{Ker} \varphi$ is provided with the quotient topology.

Proof. a) Consider $g = \sum_{i=0}^{\infty} a_i u_i \in BK_{S_T}[\{X\}]$, $g_n$ its $n$th partial sum and $\mu(i, k) = \max \{t : \theta(t, k) \leq i\}$. Since, for every $j \neq k$, $X - \beta_j = X - \beta_k + \beta_k - \beta_j$, by (20) it follows that, for every $n \geq \mu(i, k)$, the coefficient of $(X - \beta_k)^i$ in the polynomial $g_n$ written as an
element from $BK[[X - \beta_k]]$ has the form

$$c_{i,k} = \sum_{j=1}^{\mu(i,k)} P_{i,j,k}a_j,$$

(25)

where $P_{i,j,k}$ are polynomials with integral coefficients in $\beta_j$, and

$$P_{i,\mu(i,k),k} = \prod_{j=1, j \neq k}^{m(i)} (\beta_k - \beta_j)^{\theta(i,j)},$$

(26)

Then $\sum_{i=0}^{\infty} c_{i,k}(X - \beta_k)^i \in BK[[X - \beta_k]]$ and we define

$$\varphi(g) = \sum_{i=0}^{\infty} c_{i,k}(X - \beta_k)^i.$$

(27)

Since $K[X]$ is dense in $BK_S[[X]]$, for every $S$, (resp. $BK[[X - \beta_k]]$) with respect to the topology $T_S$ defined by the principal ideals $< u_i >$ (resp. the corresponding topology $T$ defined by the powers of $X - \beta_k$), $\varphi$ is continuous with respect to $T_{S_T}$ and $T$ and its restriction to $K[X]$ is the identity map, it follows that it is a $K$-algebra homomorphism. Moreover, because (25) implies that

$$\|\varphi(g)\|_{KB[[X - \beta_k]]} \leq \|g\|_{BK_S[[X]]},$$

(28)

it follows that $\varphi$ is continuous.

If $f = \sum_{i=0}^{\infty} b_i(X - \beta_k)^i \in BK[[X - \beta_k]]$, then choose $g = \sum_{i=0}^{\infty} a_i u_i \in K_{S_T}[[X]]$ such that $a_0 = b_0$ and generally by recurrence, for $i \geq 1$,

$$a_i = \left\{ \begin{array}{ll}
0, & \text{if } i \neq \mu(j,k) \text{ for every } j \\
\frac{\beta_{\mu(j,k)-1} - \sum_{s=j}^{\mu(j,k)-1} P_{j,s,k}a_s}{P_{j,\mu(j,k),k}}, & \text{if } i = \mu(j,k)
\end{array} \right.,$$

(29)

where the polynomials $P_{i,j,k}$ defined in (25) are independent of the coefficients $a_j$. Since $|\beta_i| \leq 1$, $g \in BK_{S_T}[[X]]$. If, for every $i \geq 1$, $a_i$ is given in (29), by (25) we obtain $c_{i,k} = b_i$ which implies $\varphi(g) = f$.

b) Since we may choose $g \in BK_{S_T}[[X]]$ such that $\varphi(g) = f$, by (28), we obtain, for every $x \in B^-(\beta_k, 1),$

$$|f(x) - g_n(x)| \leq \|f - g_n\|_{BK[[X - \beta_k]]} = \|\varphi(g - g_n)\|_{BK[[X - \beta_k]]} \leq \|g - g_n\|_{BK_{S_T}[[X]]}.$$

Hence it follows b).

c) Since $\varphi$ is continuous and by (29) for every $f \in BK[[X - \beta_k]]$ we may choose $g \in BK_{S_T}[[X]]$ such that $\|g\|_{BK_{S_T}[[X]]} \leq \|f\|_{BK[[X - \beta_k]]}$ by Lemma 2 from [1], p. 21 $\varphi$ is strict which implies the statement. □

By Theorem 4 we obtain easily the following result.
Corollary 2. If \( f \in BK[[X]] \), \( T = \{ \beta_j \}_{j \geq 1}, 0 \in T \), is a set of representatives of \( \bar{K} \) in \( \bar{K} \) and \( S_T = \{ \alpha_i \}_{i \geq 1} \) a sequence of elements of \( T \) such that every element of \( T \) appears infinitely many times in \( S_T \), then there exists \( g \in BK_{S_T}[[X]] \) such that \( f(x) = g(x) \), for every \( x \in B^-(0,1) \).

For \( f \in BK_{S_T}[[X]] \) denote \( Z(f) = \{ a \in \bar{K} \mid f(a) = 0 \} \) the set of all zeros of \( f \) in \( \bar{K} \) without counting the multiplicities.

Theorem 5. If \( T = \{ \beta_j \}_{j \geq 1} \) is a fixed set of representatives of \( \bar{K} \) in \( \bar{K} \), \( S_T = \{ \alpha_i \}_{i \geq 1} \) is a sequence of elements of \( T \) such that every element of \( T \) appears infinitely many times in \( S_T \), \( f \in BK_{S_T}[[X]] \) and for a fix \( j \), \( \beta_j \) is an accumulation point of \( Z(f) \), then \( B^-(\beta_j,1) \subset Z(f) \).

Proof. Suppose first \( f = \sum_{i=0}^{\infty} a_i u_i \in BK_{S_T}[[X]] \) and \( \beta_j = 0 \). Now by Theorem 4 we have a morphism \( \varphi : BK_{S_T}[[X]] \to BK[[X]] \) such that \( \varphi(f) = g \in BK[[X]] \) and \( f(x) = g(x) \), for every \( x \in B^-(0,1) \). We choose a sequence \( \{ \gamma_k \}_{k \geq 1} \) of distinct elements of \( Z(f) \cap B^-(0,1) \) such that \( \lim_{k \to \infty} \gamma_k = 0 \). Then \( f(\gamma_k) = 0 = g(\gamma_k) \), for all \( k \). We show that \( g = 0 \) in \( BK[[X]] \). If \( g \neq 0 \) and \( b_t \) is the first nonzero coefficient of \( g \) then \( g = X^t \sum_{i=0}^{\infty} b_{t+i}X^i \).

Now for \( k \) large enough \( \left| \sum_{i=0}^{\infty} b_{t+i}\gamma_k^i \right| = |b_t| \). But \( g(\gamma_k) = 0 \) implies \( b_t = 0 \). Hence \( g = 0 \) and \( f(\gamma) = g(\gamma) = 0 \) for all \( \gamma \in B^-(0,1) \). The case \( \beta_j \neq 0 \) can be reduced easily to the previous case by replacing \( X \) with \( X + \beta_j \).

Corollary 3. If \( T = \{ \beta_j \}_{j \geq 1} \), is a fixed set of representatives of \( \bar{K} \) in \( \bar{K} \) and \( S_T = \{ \alpha_i \}_{i \geq 1} \) a sequence of elements of \( T \) such that every element of \( T \) appears infinitely many times in \( S_T \), \( f \in BK_{S_T}[[X]] \) and for a fix \( j \), there exists an element \( \xi_j \in B^-(\beta_j,1) \), which is an accumulation point \( Z(f) \), then \( B^-(\beta_j,1) \subset Z(f) \).

Proof. It is enough to replace \( X \) with \( X + \beta_j - \xi_j \) and to use Corollary 2 and Theorem 5.

Now we can prove Identity Theorem for elements of \( BK_{S_T}[[X]] \).

Theorem 6. If \( T = \{ \beta_j \}_{j \geq 1} \), is a fixed set of representatives of \( \bar{K} \) in \( \bar{K} \) and \( S_T = \{ \alpha_i \}_{i \geq 1} \) a sequence of elements of \( T \) such that every element of \( T \) appears infinitely many times in \( S_T \), \( f \in BK_{S_T}[[X]] \) and for every \( j \), there exists an element \( \xi_j \in B^-(\beta_j,1) \), which is an accumulation point \( Z(f) \), then \( f(\cdot) \).

Proof. Because \( \bar{K} = \bigcup_{\beta_j \in T} B^-(\beta_j,1) \), by Corollary 3 it follows that \( f(x) = 0 \) for every \( x \in \bar{K} \). Suppose that \( f = \sum_{i=t}^{\infty} a_i u_i \neq 0 \) and \( a_t \neq 0 \). If \( u_{t+1}(X)/u_t(X) = X - \beta_j \) we choose a sequence \( \{ \gamma_k \}_{k \geq 1} \) of distinct elements of \( Z(f) \) which tends to \( \beta_j \). Since \( f(\gamma_k) = 0 \), for all \( k \) it follows that \( a_t = 0 \) which implies that \( f = 0 \).
Generalization of strictly convergent power series

Now we fixed $T = \{\beta_j\}_{j \geq 1}$ a set of representatives of $\tilde{K}$ in $\hat{K}$ and we construct a particular sequence $S_T = \{\alpha_i\}_{i \geq 1}$ of elements of $T$, such that every element of $T$ appears infinitely many times in $S_T$. Thus for every positive integer $i$ there is a unique integer $k(i)$ such that

$$\frac{(k(i)-1)k(i)}{2} < i \leq \frac{k(i)(k(i)+1)}{2}$$

(30)

and we put

$$s(i) = i - \frac{(k(i)-1)k(i)}{2}.$$  

(31)

Then we take

$$\alpha_i = \begin{cases} 
\beta_{r(i)+1}, & \text{if } \tilde{K} \text{ has } q \text{ elements} \\
\beta_{s(i)}, & \text{if } \tilde{K} \text{ is countable},
\end{cases}$$

(32)

where $r(i)$ is the remainder obtained by dividing $i$ into $q$. In this case we say that the pair $(T, S_T)$ has the standard form.

## 5 Newton analytic elements

Let $D$ be a closed subset of $C_p$. The Runge theorem of complex analysis leaded Krasner to call an analytic element a function $f : D \to C_p$ which is a uniform limit of a sequence of rational functions having no pole in $D$. By a result of Christol-Robba (see Theorem of Sec. 4.6 of [8]) it is known which series of $BC_p[[X]]$ define analytic elements. There are simple examples of series of $BC_p[[X]]$ which do not define analytic elements on $B^-(0,1)$ (see [8], p. 353).

Now we built Newton analytic elements on $B^+(0,1)$ and $B^-(0,1)$. Consider $K = C_p$, a pair $(T, S_T)$ having the standard form $D = B^+(0,1)$ and a function $f : D \to K$. We call $f$ a Newton analytic element if it is the sum of a series of $BC_{p_{S_T}}[[X]]$ on $D$. By Corollary 1 and Theorem, Sec. 4.3, Ch. 6 of [8], it follows that the Banach algebra of analytic elements on $B^+(0,1)$ is isomorphic to a subalgebra of $BC_{p_{S_T}}[[X]]$. In order to define Newton analytic elements on $B^-(0,1)$ we suppose that the pair $(T, S_T)$ has the standard form and $\beta_1 = 0$. We take $T_s \subset T$, $T_s \neq T$, such that $0 \in T_s$ and $S_{T_s}$ the sequence obtained from $S_T$ by canceling all the terms equal to $\beta_j \in T_s$. We denote by $v_i$ the corresponding polynomials defined by (3) by means of $S_{T_s}$.

In $BK_{S_T}[[X]]$ we denote by $M$ the multiplicative system generated by the polynomials $X - \beta_i$, $\beta_i \in T_s$ and $M^{-1}BK_{S_T}[[X]]$ the ring of fractions of $BK_{S_T}[[X]]$ with respect to $M$. By using an idea for power series (see [7]), we define

$$\mathcal{H}NC_{p_{S_{T_s}}}[[X]] = \{F = \sum_{i=-\infty}^{-1} a_i v_i^{-1} + f\},$$

(33)

where $a_i \in C_p$, $\lim_{i \to -\infty} a_i = 0$ and $f \in BC_{p_{S_T}}[[X]]$. If $F \in \mathcal{H}NC_{p_{S_{T_s}}}[[X]]$ we put

$$\|F\|_{\mathcal{H}NC_{p_{S_{T_s}}}[[X]]} = \max_{-\infty < i \leq -1} \{|a_i|\}, \|f\|_{BC_{p_{S_T}}[[X]]}.$$  

(34)
It can be proved that $\mathcal{HNC}_{pS_{T^c}}[[X]]$ is the completion of the algebra $M^{-1}BC_{pS_T}[[X]]$ with respect to the restriction of the norm given by (34).

Consider $K = \mathbb{C}_p$, $(T, S_T)$ a pair having the standard form with $\beta_1 = 0$, $T_s = 0$, $D = B^-(0,1)$ and a function $f : D \to \mathbb{C}_p$. We call $f$ a Newton analytic element if it is the sum of a series of $\mathcal{HNC}_{pS_{T^c}}[[X]]$.

**Remark 1.** Let $(T, S_T)$ be a pair having the standard form with $\beta_1 = 0$. If we denote the set of all Newton analytic elements on $B^+(0,1)$ as $\mathcal{H}_{N}(B^+(0,1)) = BC_{pS_T}[[X]]$ and the set of all Newton analytic elements on $B^-(0,1)$ as $\mathcal{H}_{N}(B^-(0,1)) = HNC_{pS_{T^c}}[[X]]$, with $T_s = \{0\}$, then as in the classical case that Banach $K$-algebra $\mathcal{H}_{N}(B^-(0,1))$ is isomorphic to a completion of a ring of fractions of the algebra $\mathcal{H}_{N}(B^+(0,1))$.

By Corollary 2 and Lemma 1 it follows that every $g \in BC_p[[X]]$ defines a Newton analytic element on $B^-(0,1)$ which can be extended to a Newton analytic element on $B^+(0,1)$. Hence Theorem 6 implies that for a fixed family of sequences $Z_j = \{\gamma_{j,n}\}_{n \geq 1}$, $j \geq 2$ such that $\gamma_{j,n} \in B^-(\beta_j,1)$ and each $Z_j$ has an accumulation point, every $g \in BC_{pS_T}[[X]]$ can be extended to $G \in \mathcal{H}_{N}(B^+(0,1))$ uniquely defined by its values at $\gamma_{j,n}$.

**References**


