ON PROJECTIVE COMPLEX RANDERS CHANGES

Nicoleta ALDEA 1 and Gheorghe MUNTEANU 2

Abstract

In this paper we study the relation between complex Randers changes and projective changes of complex Finsler metrics. We consider complex Randers changes of a generalized Berwald complex Finsler metric and we determine the necessary and sufficient conditions for the generalized Berwald property to be preserved by these changes. Using this theory, a recursive sequence of projectively related complex Berwald metrics is pointed out.

2000 Mathematics Subject Classification: 53B40, 53C60.
Key words: projectively related complex Finsler metrics, generalized Berwald metric, complex Berwald metric, complex Randers change.

1 Introduction

The problem of projective changes between two real Finsler metrics is quite old in geometry and it has been studied by many geometers, [8, 16, 21, 13, 19, 9, 15]. Its origin is formulated in Hilbert’s Fourth Problem: determine the metrics on an open subset in \( \mathbb{R}^n \), whose geodesics are straight lines. Two Finsler metrics, on a common underlying manifold, are called projectively related if they have the same geodesics as point sets.

The notion of Randers change has been proposed by M. Matsumoto in [16]. Further substantial contributions on this topic are due to C. Shibata [22], M. Hashiguchi, Y. Ichijyō [13], H. S. Park, I. Y. Lee [19], Bácsó, Z. Kovacs [9], etc.

The main themes from projective real Finsler geometry have recently been developed in complex Finsler geometry, [5, 6, 7]. Two complex Finsler metrics \( F \) and \( \tilde{F} \), on a common underlying manifold \( M \), are called projectively related if any complex geodesic curve, in [1]'s sense, of the first metric is also a complex geodesic curve for the second one and conversely. As we proved in [5], this means that between the spray coefficients \( G^i \) and \( \tilde{G}^i \) there is a so-called projective change \( \tilde{G}^i = G^i + B^i + P\eta^i \), where \( P \) is a smooth function on \( T'M \) with complex values and \( B^i := \frac{1}{2}(\tilde{\theta}^*i - \theta^*i) \).

1Transilvania University of Braşov, Faculty of Mathematics and Informatics Iuliu Maniu 50, Braşov 500091, Romania, e-mail: nicoleta.aldea@lycos.com
2Transilvania University of Braşov, Faculty of Mathematics and Informatics Iuliu Maniu 50, Braşov 500091, Romania, e-mail: gh.munteanu@unitbv.ro
Complex Randers metrics $\alpha + |\beta|$, where $\alpha$ is a purely Hermitian metric and $\beta$ is a $(1,0)$ - form, both on the base manifold, are remarkable in complex Finsler geometry, and they represent a situation in which Hermitian geometry properly interferes with complex Finsler geometry, [3]. A generalization of a complex Randers metric is given by $\tilde{F} = F + |\beta|$, where $(M,F)$ is a complex Finsler space, which compared to projective changes lead us to complex Randers changes, which constitute the subject of the present paper.

We consider a complex Randers change of a generalized Berwald complex Finsler metric. The complex Finsler metric obtained by a complex Randers change is also a generalized Berwald one if and only if the $(1,0)$ - form $\beta$ satisfies a special regularity condition, (see Lemma 4.1 and Corollary 4.1). The necessary and sufficient conditions for a complex Randers change to be a projective change are given in Lemma 4.2. By requiring the complex Finsler metric $F$ to be a complex Berwald one, [6], the complex Randers change $\tilde{F} = F + |\beta|$ is a projective change if and only if $\tilde{F}$ is a complex Berwald metric, (see Theorem 4.1). Moreover, by means of the obtained results we construct a recursive sequence of complex Berwald metrics which are projectively related, (Corollary 4.2).

The paper is organized as follows. In Section 2, we recall some preliminary properties of $n$-dimensional complex Finsler spaces, needed for our aforementioned study. In Section 3 we make a survey of projectively related complex Finsler metrics. Section 3 contains the proofs of the above mentioned theorems and some interesting examples.

2 Preliminaries

Let $M$ be an $n$-dimensional complex manifold and $z = (z^k)_{k=1}^n$ be the complex coordinates in a local chart. The complexified $T_C M$ of the real tangent bundle $T_R M$, splits into the sum of the holomorphic tangent bundle $T' M$ and its conjugate $T'' M$. The bundle $T' M$ is itself a complex manifold and the local coordinates in a local chart will be denoted by $u = (z^k, \eta^k)_{k=1}^n$. These are transformed into $(z'^k, \eta'^k)_{k=1}^n$ by the rules $z'^k = z^k(z)$ and $\eta'^k = \frac{\partial z'^k}{\partial z^l} \eta^l$.

A complex Finsler space is a pair $(M,F)$, where $F : T' M \to \mathbb{R}^+$ is a continuous function satisfying the following conditions:

i) $L := F^2$ is smooth on $\tilde{T'} M := T' M \setminus \{0\}$;

ii) $F(z, \eta) \geq 0$, the equality holds if and only if $\eta = 0$;

iii) $F(z, \lambda \eta) = |\lambda| F(z, \eta)$ for $\forall \lambda \in \mathbb{C}$;

iv) the Hermitian matrix $(g_{ij}(z, \eta))$ is positive definite, where $g_{ij} := \frac{\partial^2 L}{\partial \eta^i \partial \eta^j}$ is the fundamental metric tensor. Equivalently, this means that the indicatrix of the space is strongly pseudo-convex.

Consequently, from iii) we have $\frac{\partial L}{\partial \eta^i} \eta^k = \frac{\partial L}{\partial \eta^i} \bar{\eta}^k = L, \frac{\partial g_{ij}}{\partial \eta^k} \eta^k = \frac{\partial g_{ij}}{\partial \bar{\eta}^k} \bar{\eta}^k = 0$ and $L = g_{ij} \eta^i \bar{\eta}^j$.

Consider the sections of the complexified tangent bundle of $T' M$. Then by $VT' M \subset T'(T' M)$ we denote the vertical bundle, locally spanned by $\{ \frac{\partial}{\partial \eta^k} \}$, and by $VT'' M$, its conjugate. The idea of complex nonlinear connection, briefly (c.n.c.), is an instrument in the ‘linearization’ of the geometry of the manifold $T' M$. A (c.n.c.) is a supplementary
complex subbundle to $VT'M$ in $T'(T'M)$, i.e. $T'(T'M) = HT'M \oplus VT'M$. The horizontal distribution $H_u T'M$ is locally spanned by $\{ \frac{\delta}{\delta z^k} = \frac{\partial}{\partial x^s} - N^j_k \frac{\partial}{\partial \eta^j} \}$, where $N^j_k (z, \eta)$ are the coefficients of the (c.n.c.). The pair $\{ \delta_k := \frac{\delta}{\delta z^k}, \hat{\delta}_k := \frac{\partial}{\partial \eta^j} \}$ will be called the adapted frame of the (c.n.c.), which obey the change rules $\delta_k = \frac{\partial J^j_i}{\partial \xi^k} \delta^i_j$ and $\hat{\delta}_k = \frac{\partial J^j_i}{\partial \eta^k} \hat{\delta}^i_j$. By conjugation everywhere we obtain an adapted frame $\{ \delta_k, \hat{\delta}_k \}$ on $T_u''(T'M)$. The dual adapted frames are $\{ dz^k, \delta \eta^k := d\eta^k + N^k_j dz^j \}$ and $\{ dz^k, \hat{\delta} \eta^k \}$.

Let $S \in T'(T'M)$ be a complex spray. Locally, it can be expressed as follows

$$S = \eta^k \frac{\partial}{\partial z^k} - 2G^k(z, \eta) \frac{\partial}{\partial \eta^k}$$

where $G^k$ are the spray coefficients, [17], which are $(2,0)$-homogeneous with respect to $\eta$, i.e. $(\partial \eta G^i) \eta^k = 2G^i$ and $(\hat{\partial} \eta G^i) \eta^k = 0$.

Between the notions of complex spray and (c.n.c.) there exists an interdependence, each one of them determining the other, (for more details see [17]).

A (c.n.c.) related only to the fundamental function of the complex Finsler space $(M, F)$ is the so-called Chern-Finsler (c.n.c.), (see [1]), with the local coefficients $N^i_j := g^m_i \frac{\partial}{\partial z^m} \eta^j$. Subsequently, $\delta_k$ is the adapted frame with respect to the Chern-Finsler (c.n.c.). A Hermitian connection $D$, of $(1,0)$-type, which satisfies in addition $D_{JX}Z = JD_{XZ}$ for all horizontal vectors $X$, where $J$ is the natural complex structure of the manifold, is the Chern-Finsler connection, [1]. It is locally given by the following coefficients (see [17]):

$$L^i_{jk} := g^i_{jl} \delta_k g_{jl} = \hat{\delta}_j N^i_k ; C^i_{jk} := g^i_{jl} \hat{\delta}_k g_{jl}. \tag{2.2}$$

In [1]'s terminology, the complex Finsler space $(M, F)$ is Kähler iff $T^i_{jk} \eta^j = 0$ and weakly Kähler iff $g^i_{jl} T^j_{kl} \eta^j \eta^l = 0$, where $T^i_{jk} := L^i_{jk} - L^i_{kj}$. We notice that in the particular case of complex Finsler metrics which come from Hermitian metrics on $M$, called purely Hermitian metrics in [17], (i.e. $g_{ij} = g_{ij}(z)$), these two notions of Kähler are the same. On the other hand, as in Aikou’s work [2], a complex Finsler space which is Kähler and $L^i_{jk} = L^i_{kj}(z)$ is called a complex Berwald space.

In [17] is proved that the Chern-Finsler (c.n.c.) does not generally come from a complex spray. But, its local coefficients $N^i_j$ always determine a complex spray with coefficients $G^i = \frac{1}{2} N^i_j \eta^j$. Further, $G^i$ induce a (c.n.c.) denoted by $N^i_j := \dot{\delta}_j G^i$ which is called canonical in [17], and is proved that it coincides with Chern-Finsler (c.n.c.) if and only if the complex Finsler metric is Kähler. With respect to the canonical (c.n.c.), we consider the frame $\{ \delta_k, \hat{\delta}_k \}$, where $\delta_k := \frac{\partial}{\partial \xi^k} - N^j_k \hat{\delta}_j$, and its dual coframe $\{ dz^k, \hat{\delta} \eta^k \}$, where $\hat{\delta} \eta^k := d\eta^k + N^j_k dz^j$. Moreover, we associate to the canonical (c.n.c.) a complex linear connection of Berwald type $\ BF$ with its connection form

$$\omega^i_j (z, \eta) = G^i_{jk} dz^k + G^i_{jk}^c dz^k, \tag{2.3}$$

where $G^i_{jk} := \hat{\delta}_k N^i_j = G^i_{kj}$ and $G^i_{jk} := \hat{\delta}_k N^i_j$. 

On projective complex Randers changes

3
Note that the spray coefficients obey the relations $2G^i = N_j^i \eta^j = N_j^i \eta^j = G^i_{jkh} \eta^j \eta^k = L^i_{jkh} \eta^j \eta^k$. We denote by $G^i_{jkh} := \dot{\partial}_h G^i_{jk}$, $G^i_{jkh} = \dot{\partial}_h G^i_{jk}$ and $G^i_{jkh} := \dot{\partial}_h G^i_{jk}$ the $h\nu$-, $h\nu$- and $h\nu$- curvature tensors respectively; their properties are pointed out in [4].

An extension of complex Berwald spaces, directly related to the $BG$ connection, are generalized Berwald spaces, studied by us in [4]. They have the coefficients $G^i_{jk}$ depending only on the position $z$, equivalently with $\dot{\partial}_h G^i = 0$, i.e. $BG$ is of $(1, 0)$-type. Since in the Kähler case $G^i_{jk} = L^i_{jk}$, any complex Berwald space is generalized Berwald. Conversely, in [6] we proved that any generalized Berwald space, which is weakly Kähler, is a complex Berwald space.

3 Projectively related complex Finsler metrics

In Abate-Patrizio’s sense, (see [1] p. 101), the equations of a complex geodesic curve $z = z(s)$ of $(M, F)$, with $s$ a real parameter, can be expressed as follows

$$\frac{d^2 z^i}{ds^2} + 2G^i(z(s), \frac{dz}{ds}) = \theta^{*k}(z(s), \frac{dz}{ds}) ; i = \overline{1, n},$$

where by $z^i(s)$, $i = \overline{1, n}$, we denote the coordinates along of curve $z = z(s)$ and $\theta^{*k} := 2g^{ik} \delta_i^j L$. Note that $\theta^{*k}$ vanishes identically if and only if the space is weakly Kähler.

Let $\tilde{F}$ be another complex Finsler metric on the underlying manifold $M$. Corresponding to the metric $\tilde{F}$, we have the spray coefficients $\tilde{G}^i$ and the functions $\tilde{\theta}^{*k}$ are projectively related if they have the same complex geodesics as point sets. This means that for any complex geodesic curve $z = z(s)$ of $(M, F)$ there is a transformation of its parameter $s$, $\tilde{s} = \tilde{s}(s)$, with $\frac{ds}{d\tilde{s}} > 0$, such that $z = z(\tilde{s}(s))$ is a geodesic of $(M, \tilde{F})$, and conversely.

**Theorem 3.1.** ([5]). Let $F$ and $\tilde{F}$ be two complex Finsler metrics on the manifold $M$. Then $F$ and $\tilde{F}$ are projectively related if and only if there exists a smooth function $P$ on $T'M$ with complex values, such that

$$\tilde{G}^i = G^i + B^i + P \eta^i ; i = \overline{1, n},$$

where $B^i := \frac{1}{2}(\tilde{\theta}^{*k} - \theta^{*k})$.

The relations (3.2) between the spray coefficients $\tilde{G}^i$ and $G^i$ of the projectively related complex Finsler metrics $F$ and $\tilde{F}$ is called a projective change. The projective change (3.2) gives rise to various projective invariants, for more details see [6].

**Theorem 3.2.** ([5]). Let $F$ and $\tilde{F}$ be complex Finsler metrics on the manifold $M$. Then, $F$ and $\tilde{F}$ are projectively related if and only if

$$\dot{\partial}_r (\delta_k \tilde{F}) \eta^k + 2(\dot{\partial}_r G^i)(\dot{\partial}_r \tilde{F}) = \frac{1}{\tilde{F}}(\delta_k \tilde{F}) \eta^k (\dot{\partial}_r \tilde{F}) ;$$
On projective complex Randers changes

\[ B^r = -\frac{1}{\tilde{F}} \theta^{ri} (\bar{\partial}_i \tilde{F}) \bar{\eta}^r ; \quad P = \frac{1}{\tilde{F}} [ (\delta_k \tilde{F}) \eta^k + \theta^{ri} (\bar{\partial}_i \tilde{F}) ] . \] (3.3)

Moreover, the projective change is \( \tilde{G}^i = G^i + \frac{1}{\tilde{F}} (\delta_k \tilde{F}) \eta^k \eta^i \).

Note that, the weakly Kähler property is preserved by projective changes. Moreover, if the metric \( F \) is generalized Berwald, then \( \tilde{F} \) is also generalized Berwald.

4 Complex Randers changes

We consider a complex Finsler metric \( F(z, \eta) = \sqrt{g_{ij}(z, \eta) \eta^i \eta^j} \) and a differential \((1,0)\)-form \( \beta(z, \eta) := b_i(z) \eta^i \), both on the manifold \( M \).

**Definition 4.1.** A change of complex Finsler metrics \( F(z, \eta) \rightarrow \tilde{F}(z, \eta) \) is called a complex Randers change of \( F \) if

\[ \tilde{F}(z, \eta) = F(z, \eta) + |\beta| . \] (4.1)

In particular, if \( F \) is a purely Heimitian metric, i.e. \( F(z, \eta) = \sqrt{g_{ij}(z) \eta^i \eta^j} \), then \( \tilde{F}(z, \eta) \), becomes a complex Randers metric, (see Theorem 2.1, [3]). Taking into account that in the paper [5] we have an exhaustive study of the projectiveness of the complex Randers metric, our next investigation is focused on the complex Randers change with \( F \) a non purely Hermitian metric.

It is a technical computation to give the expressions of the geometric objects of the space \( (M, \tilde{F}) \), obtained by the complex Randers change (4.1). Certainly, they involve some trivial calculations which leads to

\[ \tilde{g}_{ij} = \frac{\tilde{F}}{F} g_{ij} - \frac{\tilde{F}}{2 F^3} \eta_i \eta_j + \frac{\tilde{F}}{2 |\beta|} b_i b_j + \frac{1}{2 L} \tilde{\eta}_i \tilde{\eta}_j ; \] (4.2)

\[ \tilde{g}^{\bar{i} j} = \frac{F}{\tilde{F}} g^{\bar{i} j} + |\beta| (F) |b|^2 + |\beta|) \eta^i \bar{\eta}^j - \frac{F^3}{\gamma} b^i \bar{b}^j - \frac{F}{\gamma} (\bar{\beta} \eta^i \bar{b}^j + \beta b^i \bar{\eta}^j) ; \]

\[ \tilde{N}^j_i = N^j_i + 1 \gamma (\eta_i \partial \bar{b}^r / \partial z^r - |\beta|^2 \partial b^r / \partial z^j \bar{\eta}^r) \xi^i + \frac{\beta}{2 |\beta|} k^r \partial b^r / \partial z^j , \]

where \( k^i := 2 F g^{\bar{r} i} + \frac{2 (\gamma |b|^2 + 2 |\beta|)}{\gamma} |\beta| \eta^i \bar{\eta}^r + \frac{2 F^3}{\gamma} \bar{b}^i \bar{b}^r - \frac{2 F}{\gamma} (\bar{\beta} \eta^i \bar{b}^j + \beta b^i \bar{\eta}^j) , \gamma := \tilde{F}^2 + F^2 (|b|^2 - 1) , \xi^i := \bar{\beta} \eta^i + F^2 b^i , N^j_i := g^{mk} \partial g_{n \bar{m}} / \partial z^j \eta^l , \) with the settings \n
\[ \eta_i := 2 F (\partial_i \tilde{F}) ; \quad \bar{\eta}_i := 2 \tilde{F} (\partial_i \tilde{F}) = \frac{\tilde{F}}{F} \eta_i + \frac{\tilde{F}}{F} \bar{\beta} b_i ; \] (4.3)

\[ \bar{\partial}_i |\beta| = \frac{\bar{\beta}}{2 |\beta|} b_i ; \quad b^i := g^{\bar{r} i} \bar{b}^j ; \quad |b|^2 := g^{\bar{r} i} \bar{b}^j b_j ; \quad \tilde{\bar{b}}^i := \tilde{b}^i . \]

Therefore, the spray coefficients are

\[ \tilde{G}^i = G^i + \frac{1}{2 \gamma} (\eta_i \partial \bar{b}^r / \partial z^r - |\beta|^2 \partial b^r / \partial z^j \bar{\eta}^r) \xi^j \eta^i + \frac{\beta}{4 |\beta|} k^r \partial b^r / \partial z^j \eta^i . \] (4.4)
Next, for complex Randers changes of a generalized Berwald metric we can prove the following.

**Lemma 4.1.** Let \((M, \tilde{F})\) be a connected generalized Berwald space and let \(\tilde{F}(z, \eta) = F(z, \eta) + |\beta|\) be a complex Randers change. Then, \((M, \tilde{F})\) is a generalized Berwald space if and only if \((\bar{\beta} \eta \frac{\partial \tilde{F}}{\partial z} + \beta \frac{\partial \tilde{F}}{\partial \eta})\eta^j = 0\). Moreover, given any of them, \(\tilde{G}^i = G^i\).

**Proof.** First, we prove the direct implication. If \((M, \tilde{F})\) is generalized Berwald space, then \(2\tilde{G}^i = \tilde{G}^i_{jk}(z) \eta^j \eta^k\), which means that \(\tilde{G}^i\) is quadratic in \(\eta\). Also, \(G^i\) is quadratic in \(\eta\). Thus, using (4.4) we have

\[
F|\beta|\{ - \beta[(F^2 |b|^2 + |\beta|^2)g^{ri} + |b|^2 \eta^r \eta^i - F^2 \bar{b} r b^i - \bar{\beta} \eta^i \bar{b}^r - \beta b^i \eta^r] \frac{\partial b_r}{\partial z^j} \eta^j \\
+ 4|\beta|^2(\tilde{G}^i - G^i)\} + |\beta|^2(2(F^2 |b|^2 + |\beta|^2)(\tilde{G}^i - G^i) - 2F^2 \beta g^{ri} \frac{\partial b_r}{\partial \eta} \eta^j \\
- (\bar{\beta} \eta \frac{\partial \tilde{G}^i}{\partial \eta} + \beta \frac{\partial \tilde{G}^i}{\partial \eta} \eta^i \eta^r - \bar{\beta} \eta \frac{\partial \tilde{G}^i}{\partial \eta} \eta^i \eta^r - \beta \frac{\partial \tilde{G}^i}{\partial \eta} \eta^i \eta^r b^i) = 0,
\]

which contains an irrational part and a rational part. Thus, we deduce

\[
\beta[(F^2 |b|^2 + |\beta|^2)g^{ri} + |b|^2 \eta^r \eta^i - F^2 \bar{b} r b^i - \bar{\beta} \eta^i \bar{b}^r - \beta b^i \eta^r] \frac{\partial b_r}{\partial z^j} \eta^j \\
= 4|\beta|^2(\tilde{G}^i - G^i) \quad \text{and}
\]

\[
(\bar{\beta} \eta \frac{\partial \tilde{G}^i}{\partial \eta} + \beta \frac{\partial \tilde{G}^i}{\partial \eta} \eta^i \eta^r - \bar{\beta} \eta \frac{\partial \tilde{G}^i}{\partial \eta} \eta^i \eta^r - \beta \frac{\partial \tilde{G}^i}{\partial \eta} \eta^i \eta^r b^i) + 2F^2 \beta g^{ri} \frac{\partial b_r}{\partial \eta} \eta^j
\]

\[
= 2(\alpha^2 |b|^2 + |\beta|^2)(\tilde{G}^i - G^i).
\]

Contractions with \(b_i\) and \(\eta_i\) yield

\[
(\tilde{G}^i - G^i)b_i = 0;
\]  

\[
4|\beta|^2(G^i - \tilde{G}^i)\eta_i + 2(\bar{\beta} \eta \frac{\partial \tilde{F}}{\partial \eta} \eta^i - \bar{\beta} \eta \frac{\partial \tilde{F}}{\partial \eta} \eta^i) = 0;
\]

\[
\bar{\beta}(F^2 |b|^2 + |\beta|^2)\eta_i - \beta(\alpha^2 |b|^2 + |\beta|^2) \frac{\partial b_r}{\partial z^j} \eta^j + 2 \alpha^2 |\beta|^2 \eta_i \eta^i = 0;
\]

\[
(\alpha^2 |b|^2 + |\beta|^2)(G^i - \tilde{G}^i)\eta_i + \alpha^2(\bar{\beta} \eta \frac{\partial \tilde{F}}{\partial \eta} \eta^i + \beta \frac{\partial \tilde{F}}{\partial \eta} \eta^i) \eta^i = 0.
\]

Adding the second and the third relations from (4.5), we obtain

\[
4|\beta|^2(G^i - \tilde{G}^i)\eta_i + (F^2 |b|^2 + |\beta|^2)(\bar{\beta} \eta \frac{\partial \tilde{F}}{\partial \eta} \eta^i + \beta \frac{\partial \tilde{F}}{\partial \eta} \eta^i) \eta^i = 0.
\]

This together with the fourth equation from (4.5) implies \((G^i - \tilde{G}^i)\eta_i = 0\) and \((\bar{\beta} \eta \frac{\partial \tilde{F}}{\partial \eta} + \beta \frac{\partial \tilde{F}}{\partial \eta} \eta^i) \eta^i = 0\).

Conversely, if \((\bar{\beta} \eta \frac{\partial \tilde{G}^i}{\partial \eta} + \beta \frac{\partial \tilde{G}^i}{\partial \eta} \eta^i) \eta^i = 0\), its differentiation with respect to \(\eta^m\) and the fact that \(G^i\) are holomorphic in \(\eta\), gives \((l^m \frac{\partial \tilde{G}^i}{\partial z^j} + \beta \frac{\partial \tilde{F}}{\partial \eta} \eta^i) \eta^i = 0\). The last two relations imply

\[
g^{m_i} \frac{\partial b_m}{\partial z^j} \eta^j = \beta \frac{\partial b_r}{\partial \eta} \eta^i \bar{b}^r \eta^j \eta^i \text{ and } l^m \frac{\partial b_m}{\partial \eta} \eta^i \eta^j = \beta \frac{\partial \tilde{F}}{\partial \eta} \eta^i \eta^j,
\]

which substituted into (4.4) imply \(\tilde{G}^i = G^i\) and so, \(\tilde{G}^i\) are holomorphic in \(\eta\), i.e., \(\tilde{F}\) is generalized Berwald. \(\square\)
On projective complex Randers changes

Subsequently, our aim is to determine the necessary and sufficient conditions in which the complex Randers change (4.1) is a projective change, that is, to establish when the complex Finsler metrics $F$ and $\tilde{F}$ from (4.1) are projectively related. A simple computation shows that

$$
(\delta_k \tilde{F})\eta^k = (\delta_k |\beta|)\eta^k = \frac{1}{2|\beta|}(\beta \eta^k \frac{\partial \tilde{b}_r}{\partial z^k} + \beta \frac{\partial b_r}{\partial z^k}\tilde{\eta}^r)\eta^k,
$$

(4.6)

because $(\delta_k F)\eta^k = 0$ and

$$
\theta^i(\partial_i \tilde{F}) = \frac{\tilde{\beta}}{|\beta|}(\delta_m F)b^m.
$$

(4.7)

Thanks to Lemma 4.1 we have proven,

Corollary 4.1. Let $(M, F)$ be a connected generalized Berwald space and let $\tilde{F}(z, \eta) = F(z, \eta) + |\beta|$ be a complex Randers change. Then, $(M, \tilde{F})$ is a generalized Berwald space if and only if $(\delta_k |\beta|)\eta^k = 0$.

Lemma 4.2. Let $(M, F)$ be a connected generalized Berwald space. Then, the complex Randers change (4.1) is a projective change if and only if

$$(\delta_k |\beta|)\eta^k = 0 \text{ and } B^i = -P\eta^i,$$

for any $i = \overline{1, n}$, where $P = \frac{\tilde{\beta}}{|\beta|}(\delta_m F)b^m$. Moreover, given any of them, the projective change is $\tilde{G}^i = G^i$.

Proof. Since $F$ is generalized Berwald and the metrics $F$ and $\tilde{F}$ are projectively related, then $\tilde{F}$ is also generalized Berwald. So that, by (4.6), (4.7) and Corollary 4.1, the conditions (3.3) are reduced to $B^i = -P\eta^i$, for any $i = \overline{1, n}$, where $P = \frac{\tilde{\beta}}{|\beta|}(\delta_m F)b^m$.

Conversely, since $(\delta_k |\beta|)\eta^k = 0$, then the first condition from (3.3) is identically satisfied and by (4.7), $B^i = -\frac{1}{\tilde{F}}\theta^i(\partial_i \tilde{F})\eta^i$ and $P = \frac{1}{\tilde{F}}\theta^i(\tilde{\partial}_i \tilde{F})$. All these conditions imply that the metrics $F$ and $\tilde{F}$ are projectively related.

Theorem 4.1. Let $(M, F)$ be a connected complex Berwald space. Then, the complex Randers change (4.1) is a projective change if and only if $\tilde{F}$ is a complex Berwald metric.

Proof. Suppose that the complex Randers change (4.1) is a projective change. Hence, it preserves the weakly Kähler and generalized Berwald properties of the metric $F$. Thus, $\tilde{F}$ is a complex Berwald metric.

Conversely, since $F$ and $\tilde{F}$, related by (4.1), are complex Berwald metrics, the conditions (3.3) are identically satisfied. Thus, the metrics $F$ and $\tilde{F}$ are projectively related.

An example. Let $\Delta = \{(z, w) \in \mathbb{C}^2, |w| < |z| < 1\}$ be the Hartogs triangle with the Kähler-purely Hermitian metric

$$
\alpha^i = \frac{\partial^2}{\partial z^i \partial \bar{z}^i}(\log \frac{1}{(1-|z|^2)(|z|^2-|w|^2)}) \quad \text{and} \quad \alpha^2(z, w; \eta, \theta) = a_{ij} \eta^i \eta^j,
$$

(4.8)
where $z, w, \eta, \theta$ are the local coordinates $z^1, z^2, \eta^1, \eta^2$, respectively, and $|z|^2 := z^i \bar{z}^i$, $z^i \in \{z, w\}, \eta^i \in \{\eta, \theta\}$. We choose
$$b_z = \frac{w}{|z|^2 - |w|^2}; \quad b_w = -\frac{z}{|z|^2 - |w|^2}. \tag{4.9}$$

With these tools we construct $\alpha(z, w, \eta, \theta) := \sqrt{a_{ij}(z, w) \eta^i \bar{\eta}^j}$ and $\beta(z, \eta) = b_i(z, w) \eta^i$ and from here, we obtain the complex Randers metric $F = \alpha + |\beta|$. After some calculations it follows that the spray coefficients of the metric $F$ are
$$G^z = G^z = \frac{\bar{\eta} \eta^2}{1 - |z|^2}; \tag{4.10}$$
$$G^w = G^w = \frac{w \eta^2}{z} \left( \frac{1}{1 - |z|^2} + \frac{1}{|z|^2 - |w|^2} \right) - \frac{(|z|^2 + |w|^2) \eta \theta}{z (|z|^2 - |w|^2)} + \frac{w \theta^2}{|z|^2 - |w|^2},$$
where $G^z$ and $G^w$ are the spray coefficients corresponding to the metric $\alpha$. From (4.10) and (4.9) we deduce that $F$ is a complex Berwald metric, and so, by Theorem 4.1, $\alpha$ and $F$ are projectively related.

Given the complex Berwald metric $F = \alpha + |\beta|$, (with (4.8) and (4.9)), we consider the Randers change $\tilde{F} = F + |\beta|$, where $\beta$ is same as in (4.9). Since $F$ is a complex Berwald metric and $(\delta^k |\beta|) \eta^k = 0$, we obtain $\tilde{G}^z = G^z$ and $\tilde{G}^w = G^w$, which allows us to conclude that $\tilde{F}$ is also a complex Berwald metric. Applying Theorem 4.1, the considered Randers change is projective.

We complete our considerations with the following statement.

**Corollary 4.2.** Let $(M, F_0)$ be a connected complex Berwald space. Then,
$$F_m = F_{m-1} + |\beta|, \quad m \in \mathbb{N}, \tag{4.11}$$

is a recursive sequence of complex Berwald metrics, on the complex manifold $M$, if and only if (4.11) is a projective Randers change for any $m \in \mathbb{N}$.

The proof is obtained inductively, by using Theorem 4.1.

The recursive sequence can be rewritten as $F_m = F_0 + m|\beta|, \ m \in \mathbb{N}$, and by this, we can generate some examples of complex Berwald metrics. Indeed, choosing the tools $F_0 = \alpha$ and $\beta$ from (4.8) and (4.9), we produce a lot of concrete examples of complex Berwald metrics on Hartogs triangle.

**Acknowledgment:** The first author is supported by the Sectorial Operational Program Human Resources Development (SOP HRD), financed from the European Social Fund and by Romanian Government under the Project number POSDRU/89/1.5/S/59323.

**References**

On projective complex Randers changes


