A DISTINGUISHED RIEMANNIAN GEOMETRIZATION FOR QUADRATIC HAMILTONIANS OF POLYMOMENTA

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Abstract

In this paper we construct a distinguished Riemannian geometrization on the dual 1-jet space $J^1(T, M)$ for the multi-time quadratic Hamiltonian function

$$H = h_{ab}(t)g^{ij}(t, x)p^a_ip^b_j + U^{(i)}(t, x)p^a_i + F(t, x).$$

Our geometrization includes a nonlinear connection $N$, a generalized Cartan canonical $N$-linear connection $C\Gamma(N)$ (together with its local $d$-torsions and $d$-curvatures), naturally provided by the given quadratic Hamiltonian function depending on polymomenta.

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1 Short introduction

In the last decades, numerous scientists have been preoccupied by the geometrization of Hamiltonians depending on polymomenta. In such a perspective, we point out that the Hamiltonian geometrizations are achieved in three distinct ways:

♦ the *multisymplectic Hamiltonian geometry* — developed by Gotay, Isenberg, Marsden, Montgomery and their peers (see [11], [10]);

♦ the *polysymplectic Hamiltonian geometry* — elaborated by Giachetta, Mangiarotti and Sardanashvily (see [8], [9]);
the De Donder-Weyl Hamiltonian geometry – studied by Kanatchikov (see the papers [12], [13], [14]).

In such a geometrical context, the recent studies of Atanasiu and Neagu ([4], [5], [6]) initiate the new way of distinguished Riemannian geometrization for Hamiltonians depending on polymomenta, which is in fact a natural "multi-time" extension of the already classical Hamiltonian geometry on cotangent bundles synthesized in the Miron et al.’s book [17]. Note that our distinguished Riemannian geometrization for Hamiltonians depending on polymomenta is different one by all three Hamiltonian geometrizations from above (multisymplectic, polysymplectic and De Donder-Weyl).

2 Metrical multi-time Hamilton spaces

Let us consider that \( h = (h_{ab}(t)) \) is a semi-Riemannian metric on the "multi-time" (temporal) manifold \( T^m \), where \( m = \dim T \). Let \( g = (g^{ij}(t^c, x^k, p^c_k)) \) be a symmetric d-tensor on the dual 1-jet space \( E^* = J^1(\mathbb{T}, M^n) \), which has the rank \( n = \dim M \) and a constant signature. At the same time, let us consider a smooth multi-time Hamiltonian function \( E^* \ni (t^a, x^i, p^a_i) \to H(t^a, x^i, p^a_i) \in \mathbb{R} \), which yields the fundamental vertical metrical d-tensor

\[
G^{(i)(j)}_{(a)(b)} = \frac{1}{2} \frac{\partial^2 H}{\partial p^a_i \partial p^b_j},
\]

where \( a, b = 1, \ldots, m \) and \( i, j = 1, \ldots, n \).

**Definition 1.** A multi-time Hamiltonian function \( H : E^* \to \mathbb{R} \), having the fundamental vertical metrical d-tensor of the form

\[
G^{(i)(j)}_{(a)(b)}(t^c, x^k, p^c_k) = \frac{1}{2} \frac{\partial^2 H}{\partial p^a_i \partial p^b_j} = h_{ab}(t^c)g^{ij}(t^c, x^k, p^c_k),
\]

is called a **Kronecker h-regular multi-time Hamiltonian function**.

In such a context, we can introduce the following important geometrical concept:

**Definition 2.** A pair \( MH^n_m = (E^* = J^1(\mathbb{T}, M), H) \), where \( m = \dim T \) and \( n = \dim M \), consisting of the dual 1-jet space and a Kronecker h-regular multi-time Hamiltonian function \( H : E^* \to \mathbb{R} \), is called a **multi-time Hamilton space**.

**Remark 1.** In the particular case \( (\mathbb{T}, h) = (\mathbb{R}, \delta) \), a "single-time" Hamilton space will be also called a **relativistic rheonomic Hamilton space** and it will be denoted by \( RRH^n_m = (J^1(\mathbb{R}, M), H) \).
Example 1. Let us consider the Kronecker $h$-regular multi-time Hamiltonian function $H_1 : E^* \to \mathbb{R}$ given by

$$H_1 = \frac{1}{4mc} h_{ab}(t) \varphi^i \varphi_j (x) p^a_i p^b_j,$$

where $h_{ab}(t)$ (respectively) is a semi-Riemannian metric on the temporal (spatial, respectively) manifold $T$ ($M$, respectively) having the physical meaning of gravitational potentials, and $m$ and $c$ are the known constants from Theoretical Physics representing the mass of the test body and the speed of light. Then, the multi-time Hamilton space $GMH^n_1 = (E^*, H_1)$ is called the multi-time Hamilton space of the gravitational field.

Example 2. If we consider on $E^*$ a symmetric $d$-tensor field $g^{ij}(t, x)$, having the rank $n$ and a constant signature, we can define the Kronecker $h$-regular multi-time Hamiltonian function $H_2 : E^* \to \mathbb{R}$, by setting

$$H_2 = h_{ab}(t) g^{ij}(t, x) p^a_i p^b_j + U^{(i)}_{(a)}(t, x) p^a_i + \mathcal{F}(t, x),$$

where $U^{(i)}_{(a)}(t, x)$ is a $d$-tensor field on $E^*$, and $\mathcal{F}(t, x)$ is a function on $E^*$. Then, the multi-time Hamilton space $\mathcal{NED}MH^n_2 = (E^*, H_2)$ is called the non-autonomous multi-time Hamilton space of electrodynamics. The dynamical character of the gravitational potentials $g^{ij}(t, x)$ (i.e., the dependence on the temporal coordinates $t^c$) motivated us to use the word "non-autonomous".

An important role for the subsequent development of our distinguished Riemannian geometrical theory for multi-time Hamilton spaces is represented by the following result (proved in paper [4]):

**Theorem 1.** If we have $m = \dim T \geq 2$, then the following statements are equivalent:

(i) $H$ is a Kronecker $h$-regular multi-time Hamiltonian function on $E^*$.

(ii) The multi-time Hamiltonian function $H$ reduces to a multi-time Hamiltonian function of non-autonomous electrodynamic type. In other words we have

$$H = h_{ab}(t) g^{ij}(t, x) p^a_i p^b_j + U^{(i)}_{(a)}(t, x) p^a_i + \mathcal{F}(t, x).$$

**Corollary 1.** The fundamental vertical metrical $d$-tensor of a Kronecker $h$-regular multi-time Hamiltonian function $H$ has the form

$$G^{(i)(j)}_{(a)(b)} = \frac{1}{2} \frac{\partial^2 H}{\partial p^a_i \partial p^b_j} = \begin{cases} h_{11}(t) g^{ij}(t, x^k, p^1_k), & m = \dim T = 1, \\ h_{ab}(t^c) g^{ij}(t^c, x^k), & m = \dim T \geq 2. \end{cases}$$

We recall that the transformations of coordinates on the dual 1-jet space $J^1^*(T, M)$ are given by

$$\tilde{\varphi}^a = \tilde{\varphi}^a (t^b), \quad \tilde{\varphi}^i = \tilde{\varphi}^i (x^j), \quad \tilde{p}^a_i = \frac{\partial \varphi^a_j}{\partial x^i} \frac{\partial \tilde{\varphi}^a_j}{\partial p^b_j},$$

where $\det \left( \frac{\partial \varphi^a_j}{\partial t^b} \right) \neq 0$ and $\det \left( \frac{\partial \varphi^a_j}{\partial x^j} \right) \neq 0$. In this context, let us introduce the following important geometrical concept:
**Definition 3.** A pair of local functions on \( E^* = J^{1*}(T, M) \), denoted by

\[
N = \left( N_{1}^{(a)}_{(i)b}, N_{2}^{(a)}_{(i)j} \right),
\]

whose local components obey the transformation rules

\[
\tilde{N}_{1}^{(b)}_{(i)c} \frac{\partial c}{\partial a} = N_{1}^{(c)}_{(k)a} \frac{\partial b}{\partial c} \frac{\partial k}{\partial x^j} - \frac{\partial \tilde{p}^b}{\partial x^j},
\]

\[
\tilde{N}_{2}^{(b)} \frac{\partial \tilde{x}^k}{\partial x^j} = N_{2}^{(c)} \frac{\partial b}{\partial \tilde{x}^j} - \frac{\partial \tilde{p}^b}{\partial x^j},
\]

is called a **nonlinear connection** on \( E^* \). The components \( N_{1}^{(a)}_{(i)b} \) (resp. \( N_{2}^{(a)}_{(i)j} \)) are called the **temporal** (resp. **spatial**) components of \( N \).

Following now the geometrical ideas of Miron from [15], paper [4] proves that any Kronecker \( h \)-regular multi-time Hamiltonian function \( H \) produces a natural nonlinear connection on the dual 1-jet space \( E^* \), which depends only on the given Hamiltonian function \( H \):

**Theorem 2.** The pair of local functions \( N = \left( N_{1}^{(a)}_{(i)b}, N_{2}^{(a)}_{(i)j} \right) \) on \( E^* \), where \( (\chi_{bc}^{a} \) are the Christoffel symbols of the semi-Riemannian temporal metric \( h_{ab} \))

\[
N_{1}^{(a)}_{(i)b} = \chi_{bc}^{a} \tilde{p}_{i}^{c},
\]

\[
N_{2}^{(a)}_{(i)j} = \frac{h^{ab}}{4} \left[ \frac{\partial g_{ij}}{\partial x^k} \frac{\partial \tilde{H}}{\partial \tilde{p}^b_k} - \frac{\partial g_{ij}}{\partial x^k} \frac{\partial H}{\partial \tilde{p}^b_k} + g_{ik} \frac{\partial^2 H}{\partial x^j \partial \tilde{p}^b_k} + g_{jk} \frac{\partial^2 H}{\partial x^i \partial \tilde{p}^b_k} \right],
\]

represents a nonlinear connection on \( E^* \), which is called the **canonical nonlinear connection of the multi-time Hamilton space** \( MH_{m}^{n} = (E^*, H) \).

Taking into account Theorem 1 and using the **generalized spatial Christoffel symbols** of the d-tensor \( g_{ij} \) which are given by

\[
\Gamma_{ij}^{k} = \frac{g_{kl}^{k}}{2} \left( \frac{\partial g_{li}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^j} \right),
\]

we immediately obtain the following geometrical result:

**Corollary 2.** For \( m = \dim T \geq 2 \), the canonical nonlinear connection \( N \) of a multi-time Hamilton space \( MH_{m}^{n} = (E^*, H) \), whose Hamiltonian function is given by (3), has the components

\[
N_{1}^{(a)}_{(i)b} = \chi_{bc}^{a} \tilde{p}_{i}^{c}, \quad N_{2}^{(a)}_{(i)j} = -\Gamma_{ij}^{k} \tilde{p}_{k}^{a} + T_{(i)j}^{(a)},
\]

where

\[
T_{(i)j}^{(a)} = \frac{h^{ab}}{4} \left( U_{i(b)j} + U_{j(b)i} \right), \quad (5)
\]

and

\[
U_{ib} = g_{ik} U_{(b)k}, \quad U_{kb} = \frac{\partial U_{kb}}{\partial x^r} - U_{sb} \Gamma_{kr}^{s}.
\]
3 The Cartan canonical connection $C\Gamma(N)$ of a multi-time Hamilton space

Let us consider that $MH^m_n = (J^1(T, M), H)$ is a multi-time Hamilton space, whose fundamental vertical metrical d-tensor is given by (4). Let

$$N = \left( N^{(a)}_1, N^{(a)}_2 \right)$$

be the canonical nonlinear connection of the multi-time Hamilton space $MH^m_n$.

**Theorem 3 (the generalized Cartan canonical $N$-linear connection).** On the multi-time Hamilton space $MH^m_n = (J^1(T, M), H)$, endowed with the canonical nonlinear connection $N$, there exists a unique $h$-normal $N$-linear connection

$$C\Gamma(N) = \left( \chi^{a}_{bc}, A^i_{jc}, H^i_{jk}, C^{i(k)}_{j(c)} \right),$$

having the metrical properties:

(i) $g^{ij|k} = 0$, $g^{ij|k(c)} = 0$,

(ii) $A^i_{jc} = \frac{g^{il}}{2} \frac{\delta g^{lj}}{\delta t^c}$, $H^i_{jk} = H^i_{kj}$, $C^{i(k)}_{j(c)} = C^{k(i)}_{j(c)}$,

where ”$\frac{\partial}{\partial a}$”, ”$\frac{\partial}{\partial k}$” and ”$\frac{\partial}{\partial c}$” represent the local covariant derivatives of the $h$-normal $N$-linear connection $C\Gamma(N)$.

**Proof.** Let $C\Gamma(N) = \left( \chi^{a}_{bc}, A^i_{jc}, H^i_{jk}, C^{i(k)}_{j(c)} \right)$ be an $h$-normal $N$-linear connection, whose local coefficients are defined by the relations

$$A^a_{bc} = \chi^{a}_{bc}, \quad A^i_{jc} = \frac{g^{il}}{2} \frac{\delta g^{lj}}{\delta t^c},$$

$$H^i_{jk} = \frac{g^{jr}}{2} \left( \frac{\delta g^{jr}}{\delta x^k} + \frac{\delta g^{kr}}{\delta x^j} - \frac{\delta g^{jk}}{\delta x^r} \right),$$

$$C^{k(i)}_{j(c)} = -\frac{g^{ir}}{2} \left( \frac{\partial g^{jr}}{\partial p^r_k} + \frac{\partial g^{kr}}{\partial p^r_j} - \frac{\partial g^{jk}}{\partial p^r_c} \right).$$

Taking into account the local expressions of the local covariant derivatives induced by the $h$-normal $N$-linear connection $C\Gamma(N)$, by local calculations, we deduce that $C\Gamma(N)$ satisfies conditions (i) and (ii).

Conversely, let us consider an $h$-normal $N$-linear connection

$$\tilde{C}\Gamma(N) = \left( \tilde{A}^a_{bc}, \tilde{A}^i_{jc}, \tilde{H}^i_{jk}, \tilde{C}^{i(k)}_{j(c)} \right)$$

which satisfies conditions (i) and (ii). It follows that we have

$$\tilde{A}^a_{bc} = \chi^{a}_{bc}, \quad \tilde{A}^i_{jc} = \frac{g^{il}}{2} \frac{\delta g^{lj}}{\delta t^c}.$$
Moreover, the metrical condition \( g_{ij,k} = 0 \) is equivalent with
\[
\frac{\delta g_{ij}}{\delta x^k} = g_{rj} \tilde{H}^r_{ik} + g_{ir} \tilde{H}^r_{jk}.
\]
Applying now a Christoffel process to indices \( \{i, j, k\} \), we find
\[
\tilde{H}^i_{jk} = \frac{g_{ir}}{2} \left( \frac{\delta g_{jr}}{\delta x^k} + \frac{\delta g_{kr}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^r} \right).
\]
By analogy, using the relations \( C^{i(k)}_{j(c)} = C^{k(i)}_{j(c)} \) and \( g_{ij}^{(k)} | (c) = 0 \), together with a Christoffel process applied to indices \( \{i, j, k\} \), we obtain
\[
\tilde{C}^i_{j(k)} = \frac{g_{ir}}{2} \left( \frac{\partial g_{jr}}{\partial p^c_k} + \frac{\partial g_{kr}}{\partial p^c_j} - \frac{\partial g_{jk}}{\partial p^c_r} \right).
\]
In conclusion, the uniqueness of the \textit{generalized Cartan canonical connection} \( C \Gamma(N) \) on the dual 1-jet space \( E^* = J^1(T, M) \) is clear.

\textbf{Remark 2.} (i) Replacing the canonical nonlinear connection \( N \) of the multi-time Hamilton space \( MH^m_n \) with an arbitrary nonlinear connection \( \hat{N} \), the preceding Theorem holds good.

(ii) The generalized Cartan canonical connection \( C \Gamma(N) \) of the multi-time Hamilton space \( MH^m_n \) verifies also the metrical properties
\[
h_{ab/c} = h_{abk} = h_{ab}^{(k)} = 0, \quad g_{ij/c} = 0.
\]

(iii) In the case \( m = \dim T \geq 2 \), the coefficients of the generalized Cartan canonical connection \( C \Gamma(N) \) of the multi-time Hamilton space \( MH^m_n \) reduce to
\[
A^{a}_{bc} = \chi^a_{bc}, \quad A^{i}_{jc} = \frac{g^{il}}{2} \frac{\partial g_{lj}}{\partial t^c}, \quad \tilde{H}^i_{jk} = \Gamma^i_{jk}, \quad C^{i(k)}_{j(c)} = 0.
\]

4 Local d-torsions and d-curvatures of the Cartan canonical connection \( C \Gamma(N) \)

Applying the formulas that determine the local d-torsions and d-curvatures of an \( h \)-normal \( N \)-linear connection \( D \Gamma(N) \) (see these formulas in [23]) to the generalized Cartan canonical connection \( C \Gamma(N) \), we obtain the following important geometrical results:

\textbf{Theorem 4.} The torsion tensor \( T \) of the generalized Cartan canonical connection \( C \Gamma(N) \)
of the multi-time Hamilton space $MH^m_n$ is determined by the local $d$-components

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<th>$h_M$</th>
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<tbody>
<tr>
<td>$m \geq 1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$T_{ij}^{(f)}$</td>
<td>$R_{(r)ij}^{(j)}$</td>
<td>$P_{(r)i(1)}^{(j)}$</td>
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<tr>
<td>$m = 1$</td>
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<td>0</td>
<td>0</td>
<td>$T_{a_j}$</td>
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<td>$P_{(r)a(b)}^{(f)}$</td>
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<tr>
<td>$m \geq 2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$T_{ai}$</td>
<td>$R_{(r)ai}^{(f)}$</td>
<td>$P_{(r)i(1)}^{(j)}$</td>
<td>0</td>
</tr>
<tr>
<td>$m = 2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$T_{a_j}$</td>
<td>$R_{(r)aj}^{(f)}$</td>
<td>$P_{(r)a(b)}^{(f)}$</td>
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<tr>
<td>$m \geq 2$</td>
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<td>0</td>
<td>$T_{ai}$</td>
<td>$R_{(r)ai}^{(f)}$</td>
<td>$P_{(r)i(1)}^{(j)}$</td>
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where

(i) for $m = \dim T = 1$, we have

\[
T_{ij}^{(f)} = -A_{ij}^{(r)}, \quad P_{i(1)}^{(r)} = C_{i(1)}^{r}, \quad P_{(r)1(1)}^{(j)} = \frac{\partial N_{(r)1}^{(1)}}{\partial p_{j}} + A_{r1}^{j} - \delta_{r1}^{j} \chi_{11}^{1},
\]

\[
R_{(r)ij}^{(1)} = \frac{\delta N_{(r)i}^{(1)}}{\delta x^{j}} - \frac{\delta N_{(r)j}^{(1)}}{\delta x^{i}};
\]

(ii) for $m = \dim T \geq 2$, using the equality (5) and the notations

\[
\chi_{r}^{ab} = \frac{\partial \chi_{r}^{a}}{\partial t^{b}} - \frac{\partial \chi_{r}^{b}}{\partial t^{a}} + \chi_{fa}^{d} \chi_{db}^{c} - \chi_{fb}^{d} \chi_{da}^{c},
\]

\[
\mathfrak{g}_{r}^{kij} = \frac{\partial \Gamma_{ki}^{r}}{\partial x^{j}} - \frac{\partial \Gamma_{kj}^{r}}{\partial x^{i}} + \Gamma_{ki}^{p} \Gamma_{pj}^{r} - \Gamma_{kj}^{p} \Gamma_{pi}^{r},
\]

we have

\[
T_{aj}^{(f)} = -A_{aj}^{(r)}, \quad P_{(r)a(b)}^{(f)} = \delta_{b}^{f} A_{ja}^{r}, \quad R_{(r)ab}^{(f)} = \chi_{gab}^{f} \mathfrak{g}_{r}^{g},
\]

\[
R_{(r)aj}^{(f)} = -\frac{\partial N_{(r)aj}^{(f)}}{\partial t^{a}} - \chi_{ca}^{f} T_{(r)aj}^{(c)},
\]

\[
R_{(r)ij}^{(f)} = -\mathfrak{g}_{r}^{kij} \mathfrak{g}_{r}^{k} + [T_{(r)ij}^{(f)} - T_{(r)ji}^{(f)}].
\]

**Theorem 5.** The curvature tensor $R$ of the generalized Cartan canonical connection $C\Gamma(N)$ of the multi-time Hamilton space $MH^m_n$ is determined by the following adapted
local curvature $d$-tensors:

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where, for $m \geq 2$, we have the relations

$$-R^{(d)(i)}_{(l)(a)bc} = \delta^d a \chi^{d}_{abc} - \delta^a d \chi_b^d, \quad -R^{(d)(i)}_{(l)(a)bk} = -\delta^d a \chi^d_{ik},$$

and, generally, the following formulas are true:

(i) for $m = \dim T = 1$, we have $\chi_{111} = 0$ and

$$R^{l}_{i1k} = \frac{\delta A^{l}_{i1}}{\delta x^k} - \frac{\delta H^{l}_{ik}}{\delta x^j} + A^{r}_{i1} H^{r}_{jk} - H^{r}_{ik} A^{l}_{r1} + C^{(r)}_{i(1)} R^{(1)}_{(r)1k},$$

$$R^{l}_{ijk} = \frac{\delta H^{l}_{ij}}{\delta x^k} - \frac{\delta H^{l}_{ik}}{\delta x^j} + H^{r}_{ij} H^{l}_{rk} - H^{r}_{ik} H^{l}_{rj} + C^{(r)}_{i(1)} R^{(1)}_{(r)jk},$$

$$P^{l}_{i(1)} = \frac{\partial A^{l}_{i1}}{\partial p^k} - C^{(r)}_{i(1)j} + C^{(r)}_{i(1)k} P^{r}_{(1)(1)1},$$

$$P^{l}_{ij(1)} = \frac{\partial H^{l}_{ij}}{\partial p^k} - C^{(r)}_{i(1)j} + C^{(r)}_{i(1)k} P^{r}_{(1)(1)1},$$

$$S^{l(1)}_{jk(1)(1)} = \frac{\partial C^{l(1)}_{i(1)j}}{\partial p^k} + C^{r(1)}_{i(1)j} A^{r(1)}_{i(1)k} - C^{r(1)}_{i(1)j} C^{l(1)}_{i(1)k}.$$ 

(ii) for $m = \dim T \geq 2$, we have

$$\chi^{d}_{abc} = \frac{\partial \chi^{d}_{ab}}{\partial x^c} - \frac{\partial \chi^{d}_{ac}}{\partial x^b} + \chi^{f}_{ab} \chi^{d}_{fc} - \chi^{f}_{ac} \chi^{d}_{fb},$$

$$R^{l}_{ibc} = \frac{\partial A^{l}_{ib}}{\partial x^c} - \frac{\partial A^{l}_{ic}}{\partial x^b} + A^{r}_{ib} A^{l}_{rc} - A^{r}_{ic} A^{l}_{rb},$$

$$R^{l}_{ibk} = \frac{\partial A^{l}_{ib}}{\partial x^k} - \frac{\partial A^{l}_{ik}}{\partial x^b} + A^{r}_{ib} \Gamma^{l}_{rk} - \Gamma^{r}_{ik} A^{l}_{rb},$$

$$\gamma^{l}_{ijk} = \frac{\partial \Gamma^{l}_{ij}}{\partial x^k} + \Gamma^{r}_{ij} A^{r}_{ik} - \Gamma^{r}_{ik} A^{r}_{jk}.$$ 

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