ON $\mathbb{R}$–COMPLEX FINSLER SPACES WITH KROPINA METRIC

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Abstract

In the present paper the notion of $\mathbb{R}$–complex Finsler space with Kropina metric is defined. The fundamental tensor fields $g_{ij}$ and $g_{i\bar{j}}$ are determined and the determinant and the inverse of these tensor fields are given. Also some properties of these spaces are studied. A special aproach is dedicated to the non-Hermitian $\mathbb{R}$–complex Finsler space with Kropina metric. Some examples of Hermitian and non-Hermitian $\mathbb{R}$–complex Finsler space with Kropina metric are given.

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1 $\mathbb{R}$ - complex Finsler spaces

In a previous paper [14], we extended the known definition of a complex Finsler space ([1, 2, 13, 16]), reducing the scalars to $\Lambda \in \mathbb{R}$. The outcome was a new class of Finsler space called by us the $\mathbb{R}$- complex Finsler spaces [14]. Our interest in this class of Finsler spaces issues from the fact that the Finsler geometry means, first of all, distance and this refers to curves depending on the real parameter.

In the present paper, following the ideas from real Finsler spaces with Kropina metrics ([6, 16, 10, 11, 12]), we introduce the similar notions on $\mathbb{R}$– complex Finsler spaces.

In this section we keep the general setting from [13, 14] and subsequently we recall only some needed notions.

Let $M$ be a complex manifold with $\dim \mathbb{C} M = n$, $(z^k)$ be local complex coordinates in a chart $(U, \varphi)$ and $T'M$ its holomorphic tangent bundle. It has a natural structure of complex manifold, $\dim \mathbb{C} T'M = 2n$ and the induced coordinates in a local chart on $u \in T'M$ are denoted by $u = (z^k, \eta^k)$. The changes of local coordinates in $u$ are given by the rules

$$ z'^k = z'^k (z) ; \eta'^k = \frac{\partial z'^k}{\partial z^j} \eta^j. \quad (1.1) $$

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The natural frame \( \left\{ \frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^i} \right\} \) of \( T'_u(T'M) \) transforms with the Jacobi matrix of (1.1) changes, \( \frac{\partial}{\partial z^x} = \frac{\partial z^x}{\partial z^k} \frac{\partial}{\partial z^i} + \frac{\partial z^x}{\partial \bar{z}^i} \eta^k \frac{\partial}{\partial \eta^j} \); \( \frac{\partial}{\partial \bar{z}^x} = \frac{\partial z^x}{\partial z^k} \frac{\partial}{\partial \bar{z}^i} + \frac{\partial \bar{z}^x}{\partial \eta^j} \frac{\partial}{\partial z^i} \).

A complex nonlinear connection, briefly (c.n.c.), is a supplementary distribution \( H(T'M) \) to the vertical distribution \( V(T'M) \) in \( T'(T'M) \). The vertical distribution is spanned by \( \left\{ \frac{\partial}{\partial z^i} \right\} \) and an adapted frame in \( H(T'M) \) is \( \frac{\delta}{\delta z^x} = \frac{\partial}{\partial z^i} - N^i_k \frac{\partial}{\partial \eta^j} \), where \( N^i_k \) are the coefficients of the (c.n.c.) and they have a certain rule of change at (1.1) so that \( \frac{\delta}{\delta z^x} \) transform like vectors on the base manifold \( M \) (d-tensor in [14] terminology). Next we use the abbreviations: \( \partial_k = \frac{\partial}{\partial z^k}, \delta_k = \frac{\delta}{\delta z^k}, \hat{\partial}_k = \frac{\partial}{\partial \eta^k} \) and \( \partial_k, \hat{\partial}_k, \delta_k \) for their conjugates.

The dual adapted basis of \( \left\{ \delta_k, \hat{\partial}_k \right\} \) are \( \{ dz^k, \delta \eta^k = d \eta^k + N^k_j dz^j \} \) and \( \{ d \bar{z}^k, \hat{\partial} \bar{\eta}^k \} \) theirs conjugates.

We recall that the homogeneity of the metric function of a complex Finsler space ([1, 2, 13, 16]) is with respect to all complex scalars and the metric tensor of the space, is Hermitian. From some physicist point of view, for which Hermitian condition is an impediment, it seems more appropriate that \( g_{ij} \) be an invertible metric tensor. These problems led us to in [14] to speak about Hermitian \( \mathbb{R}^- \) complex Finsler spaces (i.e. \( \det (g_{ij}) \neq 0 \)) and non-Hermitian \( \mathbb{R}^- \) complex Finsler spaces (i.e. \( \det (g_{ij}) \neq 0 \)). The present section applies our

## 2 \( \mathbb{R}^- \)–complex Finsler spaces with Kropina metric

As noticed in paper [14] an \( \mathbb{R}^- \) complex Finsler space produce two tensor fields \( g_{ij} \) and \( g_{ij} \). For a properly Hermitian geometry \( g_{ij} \) be invertible is a compulsory requirement, but from some physicist point of view, for which Hermitian condition is an impediment, it seems more appropriate that \( g_{ij} \) be an invertible metric tensor. The present section applies our
results to $\mathbb{R} -$ complex Finsler spaces with Kropina metric, better illustrating the interest for this work. As in [12] we have:

**Definition 2.1.** An $\mathbb{R} -$ complex Finsler space $(M,F)$ is called with $(\alpha, \beta)$ -metric if the fundamental function $F(z, \eta, \bar{z}, \bar{\eta})$ is $\mathbb{R} -$ homogeneous by means of functions $\alpha(z, \eta, \bar{z}, \bar{\eta})$ and $\beta(z, \eta, \bar{z}, \bar{\eta})$— depend by $\eta^i, \bar{\eta}^j, z^i, \bar{z}^j$ and $\bar{\eta}^i, (i = 1, ..., n)$ by means of $\alpha(z, \eta, \bar{z}, \bar{\eta})$ and $\beta(z, \eta, \bar{z}, \bar{\eta})$, i.e.:

\[ F(z, \eta, \bar{z}, \bar{\eta}) = F(\alpha(z, \eta, \bar{z}, \bar{\eta}), \beta(z, \eta, \bar{z}, \bar{\eta})) \]  

(2.1)

where

\[
\alpha^2(z, \eta, \bar{z}, \bar{\eta}) = \frac{1}{2} \left( a_{ij} \eta^i \eta^j + a_{ij} \bar{\eta}^i \bar{\eta}^j + 2 a_{ij} \eta^i \bar{\eta}^j \right) = \text{Re} \left\{ a_{ij} \eta^i + a_{ij} \bar{\eta}^j \right\},
\]

(2.2)

\[
\beta(z, \eta, \bar{z}, \bar{\eta}) = \frac{1}{2} \left( b_i \eta^i + b_i \bar{\eta}^i \right) = \text{Re} \left\{ b_i \eta^i \right\},
\]

with:

\[
a_{ij} = a_{ij}(z), a_{ij} = a_{ji}(z), b_i = b_i(z),
\]

(2.3)

$b_i(z)dz^i$ is a 1-form on the complex manifold $M$.

We denote:

\[
L(\alpha(z, \eta, \bar{z}, \bar{\eta}), \beta(z, \eta, \bar{z}, \bar{\eta})) = F^2(\alpha(z, \eta, \bar{z}, \bar{\eta}), \beta(z, \eta, \bar{z}, \bar{\eta})).
\]

(2.4)

**Remark 2.1** $F^2$ is a $\mathbb{R} - (\alpha, \beta)$ complex Finsler metric.

**Definition 2.2.** An $\mathbb{R} -$ complex Finsler space with $(\alpha, \beta)$-metric is called an $\mathbb{R} -$ complex Kropina space or a $\mathbb{R} -$ complex Finsler space with Kropina metric if:

\[
L(\alpha, \beta) = \left( \frac{\alpha^2}{\beta} \right)^2, \beta \neq 0.
\]

(2.4)

It follows that $F(\alpha, \beta) = \frac{\alpha^2}{\beta}, \beta \neq 0$.

Taking into account the 2-homogeneity condition of $L$:

\[
L(\alpha(z, \lambda \eta, \bar{z}, \bar{\eta}), \beta(z, \lambda \eta, \bar{z}, \bar{\eta})) = \lambda^2 L(\alpha(z, \eta, \bar{z}, \bar{\eta}), \beta(z, \eta, \bar{z}, \bar{\eta})), \lambda \in R_+,
\]

(2.5)

we have:

**Proposition 2.1.** ([5]) In an $\mathbb{R} -$ complex Finsler space with $(\alpha, \beta)$-metric the following equalities hold:

\[
\alpha L_\alpha + \beta L_\beta = 2L, \quad \alpha L_\alpha + \beta L_\alpha = L_\alpha,
\]

\[
\alpha L_\alpha + \beta L_\beta = L_\beta, \quad \alpha^2 L_\alpha + 2 \alpha \beta L_\alpha + \beta^2 L_\beta = 2L,
\]

(2.6)

where:

\[
L_\alpha := \frac{\partial L}{\partial \alpha}, \quad L_\beta = \frac{\partial L}{\partial \beta}, \quad L_\alpha \beta = \frac{\partial^2 L}{\partial \alpha \partial \beta}, \quad L_\alpha \alpha = \frac{\partial^2 L}{\partial \alpha^2}, \quad L_\beta \beta = \frac{\partial^2 L}{\partial \beta^2}.
\]

(2.6)
**Particular case:** For a $\mathbb{R}$–complex Finsler space with Kropina metric, we have:

\[
L_\alpha = \frac{4\alpha^3}{\beta^2}, \quad L_\beta = -\frac{2\alpha^4}{\beta^3}, \quad \alpha L_\alpha + \beta L_\beta = 2L \tag{2.6}''
\]

\[
L_{\alpha\alpha} = \frac{12\alpha^2}{\beta^2}, \quad L_{\alpha\beta} = -\frac{8\alpha^3}{\beta^3}, \quad L_{\beta\beta} = \frac{6\alpha^4}{\beta^4}.
\]

In the following, we propose to determine the metric tensors of an $\mathbb{R}$–complex Finsler space with Kropina metric, i.e.

\[
g_{ij} := \frac{\partial^2 L(z, \eta, \bar{z}, \lambda \bar{\eta})}{\partial \eta_i \partial \eta_j};
\]

\[
g_{i \bar{j}} := \frac{\partial^2 L(z, \eta, \bar{z}, \lambda \bar{\eta})}{\partial \eta_i \partial \bar{\eta}_j},
\]

each of these being of interest in the following.

We consider:

\[
\frac{\partial \alpha}{\partial \eta_i} = \frac{1}{2\alpha} \left( a_{ij} \eta_j + a_{i \bar{j}} \bar{\eta}_j \right) = \frac{1}{2\alpha} l_i, \quad \frac{\partial \beta}{\partial \eta_i} = \frac{1}{2\alpha} \frac{1}{2} b_i;
\]

\[
\frac{\partial \alpha}{\partial \bar{\eta}_j} = \frac{1}{2\alpha} \left( a_{i \bar{j}} \eta_i + a_{i \bar{j}} \bar{\eta}_i \right) = \frac{1}{2\alpha} l_{\bar{j}}, \quad \frac{\partial \beta}{\partial \bar{\eta}_j} = \frac{1}{2\alpha} \frac{1}{2} b_{\bar{j}},
\]

where:

\[
l_i := a_{ij} \eta_j + a_{i \bar{j}} \bar{\eta}_j, \quad l_{\bar{j}} := a_{i \bar{j}} \eta_i + a_{i \bar{j}} \bar{\eta}_i.
\]

(2.7)

We find immediately:

\[
l_i \eta_i + l_{\bar{j}} \bar{\eta}_{\bar{j}} = 2\alpha^2.
\]

(2.8)

We denote:

\[
\eta_i = \frac{\partial L}{\partial \eta_i}.
\]

(2.9)

Consequently, we obtain:

\[
\eta_i = \rho_0 l_i + \rho_1 b_i,
\]

(2.10)

where:

\[
\rho_0 = \frac{1}{2} \alpha^{-1} L_\alpha (0 \text{– homogeneity}), \quad \rho_1 = \frac{1}{2} L_\beta (1 \text{– homogeneity}),
\]

(2.10)′

Differentiating (2.10)′ by $\eta^i$ and $\bar{\eta}^j$ respectively we obtain:

\[
\frac{\partial \rho_0}{\partial \eta^i} = \rho_{-2} l_j + \rho_{-1} b_j, \quad \frac{\partial \rho_0}{\partial \bar{\eta}^j} = \rho_{-2} l_j + \rho_{-1} b_j,
\]

\[
\frac{\partial \rho_1}{\partial \eta^i} = \rho_{-1} l_j + \rho_0 b_i, \quad \frac{\partial \rho_1}{\partial \bar{\eta}^j} = \rho_{-1} l_j + \rho_0 b_i,
\]

(2.11)

where:

\[
\rho_{-2} = \frac{\alpha L_{\alpha\alpha} - L_\alpha}{4\alpha^3}, \quad \rho_{-1} = \frac{L_{\alpha\beta}}{4\alpha}, \quad \rho_0 = \frac{L_{\beta\beta}}{4}.
\]

(2.11)′

**Proposition 2.2.** The invariants of an $\mathbb{R}$–complex Finsler space with Kropina metric: $\rho_0, \rho_1, \rho_{-2}, \rho_{-1}$ are given by:

\[
\rho_0 := \frac{2F}{\beta}; \quad \rho_1 := -\frac{F^2}{\beta^2}; \quad \beta \neq 0;
\]

\[
\rho_{-2} := \frac{2}{\beta^2}; \quad \rho_{-1} := -\frac{2F^2}{\beta^2}; \quad \mu_0 := \frac{3F^2}{2\beta^2}; \quad \beta \neq 0.
\]

(2.11)″
Subscripts $-2, -1, 0, 1$ give us the degree of homogeneity of these invariants.

We have immediately:

**Proposition 2.3.** The fundamental tensor fields of an $\mathbb{R}$–complex Finsler space with Kropina metric are given by:

$$g_{ij} = \frac{2F}{\beta} a_{ij} + \frac{2}{\beta^2} l_i l_j + \frac{3F^2}{2\beta^2} b_i b_j + \frac{-2F}{\beta^2} (b_j l_i + b_i l_j).$$

(2.12)

$$g_{\bar{j}i} = \frac{2F}{\beta} \bar{a}_{ij} - \frac{2}{\beta^2} l_i l_j + \frac{F^2}{2\beta^2} b_i b_j + \frac{1}{F^2} \eta_i \eta_j,$$

(2.12')

or, in the equivalent form:

$$g_{ij} = \frac{2F}{\beta} a_{ij} - \frac{2}{\beta^2} l_i l_j + \frac{F^2}{2\beta^2} b_i b_j + \frac{1}{F^2} \eta_i \eta_j,$$

(2.13)

$$g_{\bar{j}i} = \frac{2F}{\beta} \bar{a}_{ij} - \frac{2}{\beta^2} l_i l_j + \frac{F^2}{2\beta^2} b_i b_j + \frac{1}{F^2} \eta_i \eta_j,$$

(2.13')

**Proof.** Using the relations (2.11") in Theorem 2.1 [5] by direct calculus we have the results.

The next objective is to obtain the determinant and the inverse of the tensor field $g_{ij}$.

The solution is obtained by adapting Proposition 11.2.1, p. 287 from [6] and Proposition 2.2 from [4] for an arbitrary non-singular non-Hermitian matrix $(Q_{ij})$.

The result is:

**Proposition 2.4.** Suppose:

- $(Q_{ij})$ is a non-singular $n \times n$ complex matrix with inverse $(Q^{ji})$;
- $C_i$ and $\bar{C}_i := C_{\bar{i}}$, $i = 1, \ldots, n$, are complex numbers;
- $C^i := Q^{ij} C_j$ and its conjugates; $C^2 := C^i C_i = \bar{C^i} C_i$; $H_{ij} := Q_{ij} \pm C_i C_j$

Then

i) $\det(H_{ij}) = (1 \pm C^2) \det(Q_{ij})$

ii) Whenever $1 \pm C^2 \neq 0$, the matrix $(H_{ij})$ is invertible and in this case its inverse is $H^{ji} = Q^{ji} + \frac{1}{1 \pm C^2} C^i C^j$.

**Theorem 2.1.** For a non-Hermitian $\mathbb{R}$–complex Finsler space with Kropina metric, $(L(\alpha, \beta) = \left(\frac{\alpha^2}{F^2}\right)^2, \beta \neq 0, \text{with} \ a_{ij} = 0)$, we have:

i) the contravariant tensor $g^{ji}$ of the fundamental tensor field $g_{ij}$ is:

$$g^{ji} = \frac{\beta}{2F} a^{ji} + \frac{2\beta(F^3 \omega - \beta^3)}{M} \eta^j \eta^i + \frac{F^2 \beta(4F \gamma - 3 \beta^3)}{2M} b^j b^i +$$

$$+ \frac{2\alpha^2(\beta^3 + F^2 \varepsilon)}{M} b^i \eta^j + \frac{2F \beta(\beta^3 - F^2 \delta)}{M} \eta^i b^j,$$

(2.14)
where:

\[ M = 4F^4(εδ - γω) + 4α^2β(βγ - α^2ε - α^2δ) + 3α^6ω + β^6, \]

\[ l_i = a_{ij}η^j, \quad η = \frac{2F}{β}a_{ij}η^j - \frac{F}{β^2}b_i, \]

\[ γ = a_{jk}η^jη^k = l_kη^k, \quad ε = b_jη^j, \quad ω = b_jb_j, \quad b_k = a_jk, \quad b_l = b_k, \quad \gamma = a_ii, \quad ε = 2(β^2ε - ω)^2 - \frac{q^2(2β^2ε - ω)^2}{2α^2β^2(4 + q^2A)} \]

\[ \alpha^2 - γ \]

\[ A = \frac{ωα^2 - γω + ε^2}{α^2 - γ} \]

\[ B = 2 - \frac{2βε - ω}{2α^2β^2} + \frac{α^2}{2(α^2 - γ) - \frac{α^2q^2ε^2}{2(α^2 - γ)^2(4 + q^2A)} - \frac{q^2(2βε - ω)^2}{2α^2β^2(4 + q^2A)}} \]

\[ \frac{εq^2(2βε - ω)}{β(α^2 - γ)(4 + q^2A)} \]

**Proof.** To prove the claims we apply the above Proposition in a recursive algorithm in three steps. We write \( g_{ij} \) from 2.12 in the form:

\[ g_{ij} = 2q^2(a_{ij} - \frac{1}{α}l_i l_j + \frac{q^2}{4} b_i b_j + \frac{1}{2q^2α^2}η_i η_j) \]

I. In the first step, we set \( Q_{ij} = a_{ij} \) and \( c_i = \frac{1}{2}l_i \). By applying the Proposition 2.4 we obtain \( Q_{ij}^2 = a_{ji}, c_i^2 = \frac{γ^2}{α^2}, 1 - c_i^2 = \frac{α^2 - γ}{α^2} \neq 0 \) and \( c_i^3 = \frac{η_i^j}{α} \). So, the matrix \( H_{ij} = a_{ij} - \frac{1}{α}l_i l_j \) is invertible with \( H_{ji} = a_{ji} + \frac{1}{α^2 - γ}η_i η_j \) and \( det(a_{ij} - \frac{1}{α^2}l_i l_j) = \frac{α^2 - γ}{α^2} det(a_{ij}). \) II. Now, we consider \( Q_{ij} = a_{ij} - \frac{1}{α^2}l_i l_j \) and \( c_i = \frac{q^2}{4} b_i b_j \). By applying the Proposition 2.4 we obtain this time: \( Q_{ij} = a_{ji} + \frac{1}{α^2 - γ}η_i η_j, c_i^2 = \frac{q^2}{4} (ω + \frac{1}{α^2 - γ}ε^2), 1 + c_i^2 = 1 + \frac{q^2(α^2ω - γω + ε^2)}{4(α^2 - γ)} = \frac{4 + q^2A}{4} \neq 0 \), where \( A = \frac{α^2ω - γω + ε^2}{α^2 - γ} \) and \( c_i^3 = \frac{q^2}{2}(b_i^2 + \frac{ε^2}{α^2 - γ} η_i^j) \).

It results that the inverse of \( H_{ij} = a_{ij} - \frac{1}{α^2}l_i l_j + \frac{q^2}{4} b_i b_j \) exists and it is \( H_{ji} = a_{ji} + \frac{1}{α^2 - γ}(1 - \frac{q^2ε}{4 + q^2A}(α^2 - γ))η_i η_j - \frac{q^2}{4 + q^2A}b_i b_j - \frac{q^2ε}{(4 + q^2A)(α^2 - γ)}(η_i η_j + b_i b_j) \) and

\[ det(a_{ij} - \frac{1}{α^2}l_i l_j + \frac{q^2}{4} b_i b_j) = \frac{4 + q^2A}{4} α^2 - γ det(a_{ij}). \]

III. Finally, we put \( Q_{ij} = a_{ij} - \frac{1}{α^2}l_i l_j + \frac{q^2}{4} b_i b_j \) and \( c_i = \frac{1}{\sqrt{2αq^2}} η_i \).

From here, we obtain:
\[
Q^{ij} = a^{ij} + \frac{1}{\alpha^2 - \gamma} (1 - \frac{q^2 \varepsilon^2}{(4 + q^2 A)(\alpha^2 - \gamma)}) \eta^i \eta^j - \frac{q^2}{4 + q^2 A} b^i b^j - \frac{q^2 \varepsilon}{(4 + q^2 A)(\alpha^2 - \gamma)} (\eta^i b^j + b^i \eta^j),
\]

\[
c^2 = \left[ a^{ij} + \frac{1}{\alpha^2 - \gamma} (1 - \frac{q^2 \varepsilon^2}{(4 + q^2 A)(\alpha^2 - \gamma)}) \eta^i \eta^j - \frac{q^2}{4 + q^2 A} b^i b^j - \frac{q^2 \varepsilon}{(4 + q^2 A)(\alpha^2 - \gamma)} (\eta^i b^j + b^i \eta^j) \right] \frac{1}{2\alpha^2 q^2} \eta^i \eta^j
\]

\[
1 + c^2 = 2 - \frac{2\beta \varepsilon - \omega}{2\alpha \beta \gamma} + \frac{\alpha^2}{\alpha \beta \gamma} \frac{(2\varepsilon - \omega)^2}{(4 + q^2 A)(\alpha^2 - \gamma)} (2\varepsilon - \omega),
\]

\[
\frac{\varepsilon q^2 (2\beta \varepsilon - \omega)^2}{\beta (\alpha^2 - \gamma)(4 + q^2 A)} \neq 0
\]

and

\[
c^i = \frac{1}{\sqrt{2\alpha q^2}} (M \eta^i + Nb^i),
\]

where

\[
M = \frac{\sqrt{2}}{\alpha} + \frac{\alpha}{\sqrt{2}(\alpha^2 - \gamma)} (1 - \frac{q^2 \varepsilon^2}{(4 + q^2 A)(\alpha^2 - \gamma)}) - \frac{q^4 \varepsilon^2 (2\beta \varepsilon - \omega)}{\sqrt{2} \beta (4 + q^2 A)(\alpha^2 - \gamma)} \quad \text{and}
\]

\[
N = -\frac{1}{\sqrt{2\alpha \beta}} \frac{q^2 (2\beta \varepsilon - \omega)}{\sqrt{2}(4 + q^2 A)\beta \alpha} \frac{\alpha \varepsilon q^2}{\sqrt{2}(4 + q^2 A)(\alpha^2 - \gamma)}.
\]

By applying the Proposition 2.4 we obtain that the inverse of

\[
H_{ij} = a_{ij} - \frac{1}{\alpha^2} l_i l_j + \frac{q^2}{4} b_i b_j + \frac{1}{2\alpha^2 q^2} \eta_i \eta_j
\]

is

\[
H^{ji} = a^{ji} + \frac{1}{\alpha^2 - \gamma} (1 - \frac{q^2 \varepsilon^2}{(4 + q^2 A)(\alpha^2 - \gamma)}) \eta^j \eta^i - \frac{q^2}{4 + q^2 A} b^j b^i - \frac{q^2 \varepsilon}{(4 + q^2 A)(\alpha^2 - \gamma)} (\eta^j b^i + b^j \eta^i) - \frac{1}{2\alpha^2 q^2} (M \eta^i + Nb^i) (M \eta^j + Nb^j)
\]

and \(\text{det}(a_{ij} - \frac{1}{\alpha^2} l_i l_j + \frac{q^2}{4} b_i b_j + \frac{1}{2\alpha^2 q^2} \eta_i \eta_j) = B \frac{4 + q^2 A}{4} \alpha^2 - \gamma \alpha^2 \text{det}(a_{ij})\), where \(B = 1 + c^2\). But, \(g_{ij} = 2q^2 H_{ij}\), with \(H_{ij}\) from III. Thus, \(g^{ij} = \frac{1}{2q^2} H^{ji}\) and \(\text{det}(g_{ij}) = (2q^2)^n B \frac{4 + q^2 A}{4} \alpha^2 - \gamma \alpha^2 \text{det}(a_{ij})\). From here, immediately results i) and ii).
Proposition 2.5. In a non-Hermitian $\mathbb{R}$-complex Finsler space with Kropina metric we have the following properties:

\[ \gamma + \gamma = l_1 \eta^i + l_2 \eta^j = a_{ij} \eta^j \eta^i + a_{jk} \bar{\eta}^j \eta^k = 2\alpha^2 \]  

(2.16)

\[ \epsilon + \bar{\epsilon} = b_j \eta^j + b_j \bar{\eta}^j = 2\beta, \delta = \epsilon, \]  

(2.17)

where:

\[ l_i = a_{ij} \eta^j, \eta_{ij} = \frac{2F}{\beta} a_{ij} \eta^j - \frac{2F}{\beta} b_i, \gamma = a_{jk} \bar{\eta}^j \eta^k = l_k \eta^k, \epsilon = b_j \eta^j, \omega = b_j b^j, \]

\[ b^j = a_{jk} b_j, b_j = b^k a_{kl}, \delta = a_{jk} \eta^j b^k = l_k b^k, l^j = a^j i l_i = \eta^j. \]

Example 1. We set $\alpha$ as

\[ \alpha^2(z, \eta) := \frac{(1 + \epsilon |z|^2) \sum_{k=1}^n \Re(\eta^k)^2 - \epsilon \Re <z, \eta>^2}{(1 + \epsilon |z|^2)^2}, \]  

(2.18)

where $|z|^2 := \sum_{k=1}^n z^k \bar{z}^k$, $<z, \eta> := \sum_{k=1}^n z^k \bar{\eta}^k$, defined over the disk $\Delta^\epsilon = \{ z \in \mathbb{C}^n, |z| < \epsilon \}$, if $\epsilon < 0$, on $\mathbb{C}^n$ if $\epsilon = 0$ and on the complex projective space $P^\epsilon(\mathbb{C})$ if $\epsilon > 0$. By computation, we obtain $a_{ij} = \frac{1}{1 + \epsilon |z|^2} \left( \delta_{ij} - \epsilon \frac{\bar{z}^i \bar{z}^j}{1 + \epsilon |z|^2} \right)$ and $a_{ij} = 0$ and so, $a^2(z, \eta) = \frac{1}{2} \left( a_{ij} \eta^i \eta^j + a_{ij} \bar{\eta}^i \bar{\eta}^j \right)$. Now, taking $\beta(z, \eta) := \frac{\Re <z, \eta>}{1 + \epsilon |z|^2}$, we obtain some examples of non-Hermitian $\mathbb{R}$-complex Kropina metrics:

\[ F_\epsilon := \frac{(1 + \epsilon |z|^2) \sum_{k=1}^n \Re(\eta^k)^2 - \epsilon \Re <z, \eta>^2}{(1 + \epsilon |z|^2)^2}. \]

(2.19)

Theorem 2.2. For a Hermitian $\mathbb{R}$-complex Finsler space with Kropina metric ($F = \frac{a^2}{\beta}$, $\beta \neq 0$, $a_{ij} = 0$) we have:

\[ g^{jk} = \frac{\beta(4\beta \omega - \beta(4\beta \omega - 2F \beta + \omega))}{2F \beta} \eta^j \eta^k + \frac{\beta(4\beta \omega - 2F \beta + \omega)}{2F \beta} b^j \eta^k + \frac{\beta(4\beta \omega - 2F \beta + \omega)}{2F \beta} b^j b^k, \]  

(2.20)

where:

\[ N = |\epsilon|^2 - \alpha^2 \omega + 3F \beta \omega + 8\beta^2 - 8 \beta \Re(\epsilon) \]

\[ \alpha^2 = a_{ij} \eta^i \eta^j = l_i \eta^i, l^j = a^j i l_i = \eta^j \]

(2.21)

\[ b^k = a_{jk} b_j, \epsilon = b^k \bar{b}^k, \omega = b^j b^j. \]

Proof. Assuming $a_{ij} = 0$, from (2.7)', (2.10) and Proposition 2.2 it follows:

\[ l_i = a_{ij} \bar{\eta}^j, \eta_i = \frac{2F}{\beta} a_{ij} \eta^j - \frac{F^2}{\beta} b_i \]
Considering:
\[ \tilde{g}^{jk} = \frac{\beta}{2F} \tilde{a}^{jk} + C^* \tilde{\eta}^j \tilde{\eta}^k + D^* \tilde{b} \tilde{b}^k + E^* \tilde{b} \eta^k + F^* \eta \tilde{b}^k \]
and on the condition:
\[ g_{ij} \tilde{g}^{jk} = \delta^k_i, \]
by direct calculus we obtain:
\[ C^* = \beta \left( -4 \beta + F \omega \right) 
\frac{2F^2}{N}, D^* = \beta \left( -3 \beta + \alpha^2 \right) 
\frac{2F}{N}, E^* = \beta \left( 4 \beta - \bar{\varepsilon} \right) 
\frac{2F}{N}, F^* = \beta \left( 4 \beta - \varepsilon \right) 
\frac{2F}{N} \]
where \( N \) is given in (2.21)

Example 2. We consider \( \alpha \) given by
\[ \alpha^2(z, \eta) := \frac{|\eta|^2 + \varepsilon \left( |z|^2 |\eta|^2 - |<z, \eta>|^2 \right)}{(1 + \varepsilon |z|^2)^2}, \quad (2.22) \]
defined over the disk \( \Delta^n_r = \{ z \in \mathbb{C}^n, |z| < r, \quad r := \sqrt{\frac{1}{|\varepsilon|}} \} \) if \( \varepsilon < 0 \), on \( \mathbb{C}^n \) if \( \varepsilon = 0 \) and on the complex projective space \( P^n(\mathbb{C}) \) if \( \varepsilon > 0 \), where \( |<z, \eta>|^2 := |<z, \eta> < z, \eta> \).

By computation, we obtain \( a_{ij} = 0 \) and \( a_{ij} = \frac{1}{1+\varepsilon |z|^2} \left( \delta_{ij} - \varepsilon \frac{\tilde{z}^j \tilde{z}^i}{1+\varepsilon |z|^2} \right) \) and so, \( \alpha^2(z, \eta) = a_{ij}(z) \eta^i \tilde{\eta}^j \). Thus it determines purely Hermitian metrics which have special properties. They are Kähler with constant holomorphic curvature \( K_\alpha = 4\varepsilon \). Particularly, for \( \varepsilon = -1 \) we obtain the Bergman metric on the unit disk \( \Delta^n := \Delta^n_{1}; \) for \( \varepsilon = 0 \) the Euclidean metric on \( \mathbb{C}^n \), and for \( \varepsilon = 1 \) the Fubini-Study metric on \( P^n(\mathbb{C}) \). Setting \( \beta(z, \eta) \) as in Example 1, we obtain some examples of Hermitian \( \mathbb{R} \)-complex Kropina metrics:
\[ F_\varepsilon := \frac{|\eta|^2 + \varepsilon \left( |z|^2 |\eta|^2 - |<z, \eta>|^2 \right)}{(1+\varepsilon |z|^2)^2} \frac{Re <z, \eta>}{1+\varepsilon |z|^2}. \quad (2.23) \]

References


