ON SOME CONDITIONS FOR UNIVALENCE

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Abstract

We present some sufficient conditions for univalence in terms of the coefficients of an analytic functions.

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1 Introduction

Let $A$ be the class of analytic functions $f$ in the unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$ of the form

$$f(z) = z + a_2 z^2 + \ldots + a_n z^n \ldots , \quad z \in U$$ (1)

Let $S$ denote the class of functions $f \in A$, $f$ univalent in $U$. The usual subclasses of $S$ consisting of starlike, convex and uniformly convex functions will be denoted by $ST$, $CV$ and respectively $UCV$.

Given the sequence of coefficients $(a_n)$ in (1), how does this sequence influence the geometric properties of $f$ and can we decide if $f$ is univalent in $U$? So, it is well-known that if $f$ is given by (1) and

$$\sum_{n=2}^{\infty} n |a_n| \leq 1,$$

then $f$ is univalent in $U$. The same condition assures that $f$ is a starlike function. (see[1]).

In [2] Goodman gave the sufficient condition

$$\sum_{n=2}^{\infty} 3n(n-1) |a_n| \leq 1,$$

for the function $f$ of the form (1) to be uniformly convex. An improvement of this condition was obtained in [5]. If

$$\sum_{n=2}^{\infty} n(2n-1) |a_n| \leq 1,$$

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then the function $f$ of the form (1) is in $UCV$.

The above results are related to the univalence of an analytic function $f$ in $U$. We are interesting if similar conditions can assure the analyticity and the univalence of a family of functions defined by an integral operator. Our considerations are based on the following results.

2 Preliminaries

Theorem 1. ([6]). Let $f \in A$, $\alpha \in \mathbb{C}$, $|\alpha - 1| < 1$. If for all $z \in U$
\[ |f'(z) - 1| < 1, \]
then the function
\[ F_\alpha(z) = \left( \alpha \int_0^z u^{\alpha-1} f'(u) \, du \right)^{1/\alpha} \]
is analytic and univalent in $U$, where the principal branch is intended.

Theorem 2. ([3]). Let $f \in A$, $\alpha \in \mathbb{C}$, $\text{Re} \alpha \geq 1$. If the inequality
\[ \left| zf'(z) f(z) - 1 \right| < 1 \]
(4)
is true for all $z \in U$, then the function $F_\alpha$ defined by (3) is analytic and univalent in $U$.

Theorem 3. ([4]). Let $f \in A$, $\beta \in \mathbb{C}$, $\text{Re} \beta > 0$. If
\[ \left| \frac{zf''(z)}{f'(z)} \right| \leq 1, \]
(5)
for all $z \in U$, then for all complex numbers $\alpha$, $\text{Re} \alpha \geq \text{Re} \beta$, the function $F_\alpha$ defined by (3) is analytic and univalent in $U$.

3 Main results

Theorem 4. Let $f \in A$, $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $z \in U$. If
\[ \sum_{n=2}^{\infty} n |a_n| < 1, \]
(6)
then $f$ is univalent in $U$ and for all $\alpha \in \mathbb{C}$, $|\alpha - 1| < 1$, the functions
\[ F_\alpha(z) = z \cdot \left[ 1 + \sum_{n=2}^{\infty} \frac{n \alpha a_n}{\alpha + n - 1} z^{n-1} \right]^{1/\alpha} \]
is analytic and univalent in $U$. 
Proof. For all \( z \in U \), the condition (2) of Theorem 1 is verified.

\[
|f'(z) - 1| = \left| \sum_{n=2}^{\infty} n a_n z^{n-1} \right| \leq \sum_{n=2}^{\infty} n |a_n| < 1.
\]

Thus \( f(z) = F_1(z) \) is univalent and for every \( \alpha \in \mathbb{C}, |\alpha - 1| < 1 \), the functions \( F_\alpha \) defined by (7) are analytic and univalent in \( U \).

**Theorem 5.** Let \( f \in A, f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ z \in U \). If

\[
\sum_{n=2}^{\infty} n |a_n| < 1, \tag{8}
\]

then \( f \) is starlike in \( U \) and for all \( \alpha \in \mathbb{C}, \text{Re}\alpha \geq 1 \), the functions \( F_\alpha \) defined by (7) are analytic and univalent in \( U \).

**Proof.** It is easy to verify that the assumption (4) of Theorem 2 is satisfied. If (8) holds, then \( \sum_{n=2}^{\infty} n |a_n| < 1 \) and it follows

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| = \left| \frac{a_2 z + \ldots + (n-1)a_n z^{n-1} + \ldots}{1 + a_2 z + \ldots + a_n z^{n-1} + \ldots} \right| \leq \frac{\sum_{n=2}^{\infty} (n-1) |a_n|}{1 - \sum_{n=2}^{\infty} |a_n|}
\]

The last expression is bounded above by 1 if \( \sum_{n=2}^{\infty} n |a_n| < 1 \). Since (4) implies \( \text{Re} \frac{zf'(z)}{f(z)} > 0 \) we deduce that \( f \) is starlike in \( U \) and in view of Theorem 2, the functions \( F_\alpha \) are analytic and univalent in \( U \).

**Theorem 6.** Let \( f \in A, f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ z \in U \). If

\[
\sum_{n=2}^{\infty} n(n-1) |a_n| < \frac{27 - 6\sqrt{3}}{23} \approx 0.722, \tag{9}
\]

then \( f \) is univalent in \( U \) and for all \( \alpha \in \mathbb{C}, \text{Re}\alpha \geq 1 \), the functions \( F_\alpha \) defined by (7) are analytic and univalent in \( U \).

**Proof.** First, we note that Theorem 3 improves Becker’s univalence criterion. Indeed, for \( \beta = 1 \), the condition (5) becomes

\[
(1 - |z|^2) \left| \frac{zf''(z)}{f'(z)} \right| \leq 1, \quad z \in U
\]

and assures the univalence of the function \( f \) and also of the functions \( F_\alpha \) defined by (3), for all \( \alpha \in \mathbb{C}, \text{Re}\alpha \geq 1 \). We consider now the function \( h : [0, 1] \rightarrow \mathbb{R}, h(x) = x(1 - x^2) \) which has a maximum value in the point \( x_0 = \sqrt{3}/3 \), namely

\[
0 < h(x) \leq \frac{2\sqrt{3}}{9}, \quad x \in [0, 1].
\]
It follows that
\[
(1 - |z|^2) \left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{2\sqrt{3}}{9} \cdot \max_{z \in U} \left| \frac{f''(z)}{f'(z)} \right| \leq 1,
\]
for
\[
\left| \frac{f''(z)}{f'(z)} \right| \leq \frac{3\sqrt{3}}{2} \quad z \in U. \tag{10}
\]
Suppose that \(\sum_{n=2}^{\infty} n(n-1)|a_n| \leq v < 1\). Then \(\sum_{n=2}^{\infty} n |a_n| < v\) and
\[
\frac{1}{1 - \sum_{n=2}^{\infty} n|a_n|} < \frac{1}{1 - v}
\]
For all \(z \in U\) we have
\[
\left| \frac{f''(z)}{f'(z)} \right| \leq \frac{\sum_{n=2}^{\infty} n(n-1)|a_n|}{1 - \sum_{n=2}^{\infty} n|a_n|}
\]
Therefore, the inequality (10) is satisfied if
\[
\sum_{n=2}^{\infty} n(n-1)|a_n| < \frac{3\sqrt{3}}{2 + 3\sqrt{3}} = \frac{27 - 6\sqrt{3}}{23}.
\]
Thus, in view of Theorem 3, for all \(\alpha \in \mathbb{C}\), \(\Re \alpha \geq 1\), the functions \(F_\alpha\) defined by (7) are analytic and univalent in \(U\).

The following result improves the bounded (9) given in Theorem 6.

**Theorem 7.** Let \(f \in A\), \(f(z) = z + \sum_{n=2}^{\infty} a_n z^n\), \(z \in U\). If
\[
\sum_{n=2}^{\infty} n(2n + 3\sqrt{3} - 2) |a_n| < 3\sqrt{3} \tag{11}
\]
then \(f\) is univalent in \(U\) and for all \(\alpha \in \mathbb{C}\), \(\Re \alpha \geq 1\), the functions \(F_\alpha\) defined by (7) are analytic and univalent in \(U\).

**Proof.** In view of Theorem 6, we have
\[
\left| \frac{f''(z)}{f'(z)} \right| \leq \frac{\sum_{n=2}^{\infty} n(n-1)|a_n|}{1 - \sum_{n=2}^{\infty} n|a_n|}
\]
The last expression is bounded above by \(3\sqrt{3}/2\) if \(\sum_{n=2}^{\infty} n(2n + 3\sqrt{3} - 2) |a_n| < 3\sqrt{3}\). □

Taking into account the result of paper [5], we can give the following

**Corollary 1.** If \(\sum_{n=2}^{\infty} n(2n + 3\sqrt{3} - 2) |a_n| \leq 1\), then the function \(f\) of the form (1) is in \(UCV\) and for all \(\alpha \in \mathbb{C}\), \(\Re \alpha \geq 1\), the functions \(F_\alpha\) defined by (7) are analytic and univalent in \(U\).
References


